A.A. 2022/23 - Mathematical Physics
Control Systems Engineering

Self-evaluation questions and exercises, October 23 2022

- Define solutions and orbits for a differential equation.
- What is a vector field on an open set of $\mathbb{R}^n$? What does it mean that the vector field is complete?
- Define equilibria of a vector field. How can you dynamically characterize equilibria?
- How do you construct the phase portrait of $\dot{z} = X(z)$, $z \in \mathbb{R}$?
- What is the "Allee effect" in population dynamics? Can you write a model including this effect in the logistic equation?
- Explain consequences of Existence and Uniqueness Theorem on orbits of differential equations.
- What can you say about a point x^* in the phase space if there is a solution of a differential equations converging to x^* for $t \to +\infty$ (or $-\infty$)?
- Be sure to draw all the phase portraits studied during the lectures.
- Enumerate the properties of the flow for a vector field.
- What is the linearization of a vector field around an equilibrium?
- What does it mean that a second order differential equation is equivalent to a system of first order differential equations?
- For a second order differential equation, write the general form of equilibria and explain the difference between equilibrium and equilibrium configuration.
- Let $k \in \mathbb{R}$. Draw the phase portrait of $\dot{z} = z^2 + k$, $z \in \mathbb{R}$, for $k < 0$, $k = 0$ and $k < 0$. In each case, compare the phase portrait near every equilibrium with the one of the linearization around the equilibrium.
- Determine equilibria of the system

  $$\begin{cases} \dot{x} = x - 2y + xy \\ \dot{y} = 2x + y - y^2 \end{cases}$$

  and linearize the system around one of the equilibria.

- Determine equilibria of the system

$$\begin{cases} \dot{x} = 2(x-1) + y + (x-1)y \\ \dot{y} = 1 - x + 2y + (x-1)y^2 \end{cases}$$

  and linearize the system around one of the equilibria.

- Is it possible that the differential equation $\dot{z} = X(z)$ with $z \in \mathbb{R}$ has periodic solutions? Justify the answer.

- Claim and proof of the theorem on the exponential divergence of solutions. Is this estimate "optimal"?

- What does it mean "sensitive dependence" on initial data? Introduce some models showing this behavior.

- How solutions of a differential equations depend on parameters?

- Define the matrix exponential.

- Be sure to deduce the phase portraits of linear systems on the plane, with diagonalizable matrix.

- Be sure to remind and interpret the bifurcation diagram of linear systems on the plane, with diagonalizable matrix.

- Draw the phase portrait of the linear system associated to the matrix $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$.

- Draw the phase portraits of the linear systems associated to

$$\begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}.$$

  In the case of nodes and saddles, determine also the stable and unstable eigenspaces.

- Define elliptic and hyperbolic equilibria. What can we say for the phase portrai, locally around these equilibria, for a non-linear differential equation?

- Define first integrals for a differential equation.

- Define the Lie derivative of a function f along a vector field X.

- What does it mean that $c \in \mathbb{R}$ is a regular value for a function f? Explain consequences of this property on the corresponding level set.
- Define (topological) stability, unstability and asymptotic stability for an equilibrium.
- Claim and prove Lyapunov Theorems on simple and asymptotic stability.
- How do you draw the phase portrait for a 1-dim Newton equation $m\ddot{x} = -V'(x)$? Where energy levels are regular manifolds? How these levels intersect the x-axis? Explain in details.
- Be sure to deduce all the phase portraits of 1-dim Newton equations $m\ddot{x} = -V'(x)$ studied during lectures.
- Let $f : \mathbb{R} \to \mathbb{R}$. Which is the difference between the phase portrait of

  $$\dot{z} = f(z), \qquad z \in \mathbb{R}$$

  and the phase portrait of

  $$\ddot{x} = f(x), \qquad x \in \mathbb{R}?$$

- Let consider the system

  $$\begin{cases} \dot{x} = x(1 - x - y) \\ \dot{y} = y(2 - x - y) \end{cases}$$

  1. Determine equilibria, linearize the system around each equilibrium and draw the phase portraits of the linearized systems.
  2. In such a case, is it possible to deduce the phase portrait of the original system?

- Describe qualitatively the motion of a point of mass m subjected to a 1-dim conservative force with potential:

  $$V(x) = (2x^2 - x)e^{-x/2}$$

  or

  $$V(x) = x \sin x.$$

- Let consider a point of mass $m = 1$ subjected to a 1-dim conservative force with potential $V(x) = -(x^2 - 1)^2$. For the energy level $E = 0$, calculate the time needed to go from $x_0 = 0$ to $x_1 = 1$, assuming that the initial velocity is positive.

- What does it mean that a dynamical system has a bifurcation?

- Other interesting exercises, you can have a curious look to:
  From 3.1.1 to 3.1.5 (pages 79-80), from 3.2.1 to 3.2.4 (page 80), from 3.4.1 to 3.4.10 (pages 82-83), from 5.1.1 to 5.1.9 (pages 140-141), from 5.2.1 to 5.2.10 (pages 142-143), Example 6.5.2 in "Nonlinear Dynamics and Chaos", S.H. Strogatz.