MATH4U - 20/10/2022

Sunday, 16 October 2022 11:54

1) QUESTIONS ABOUT PREVIOUS EXERCISES

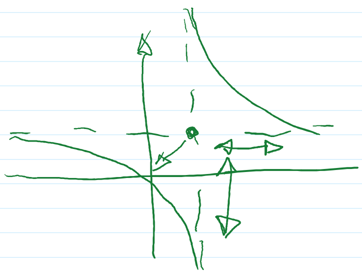
2) EQUATIONS WITH COMPLEX NUMBERS

4) $\underbrace{\log\left(\frac{1-3x}{x+2}\right)} \to$ find K s.t. $\log\left(\frac{1-3x}{Kx+2}\right)$ is odd/even function

$\to \frac{1-3x}{x+2} > 0$

e.g. $\exists K$ s.t. $f(0)=0$ ?

$f(x) = \pm f(-x)$



10) $\sum_{K=0}^{n} \frac{1}{K^2} < 2$ hint: verify that $< 2 - \frac{1}{n} < 2$

COMPLEX NUMBER

12) $z^2 - \frac{1}{4}(z-\bar{z})^2 - 4 = 8i$

$z = x+iy \to \cancel{x}+iy-(\cancel{x}-iy) = 2iy$

$(x+iy)^2 - \frac{1}{4}(2iy)^2 = 4+8i$

$x^2 \cancel{-y^2} + 2ixy - \cancel{\frac{1}{4}(-4y^2)} = 4+8i$

$x^2 + 2ixy = 4+8i$

$\begin{cases} Re(I) = Re(II) \\ Im(I) = Im(II) \end{cases} \to \begin{cases} x^2 = 4 \\ 2xy = 8 \end{cases}$ $\quad x = \pm 2 \quad y = \frac{4}{x} = \pm 2$

$\{z \in \mathbb{C} : z = 2+2i, -2-2i\}$

13) $\bar{z}^3 z^4 = -2z^2$

$z = \rho e^{i\theta}$

$\rho^3 e^{-i3\theta} \cdot \rho^4 e^{i4\theta} = -2\rho^2 e^{i2\theta}$

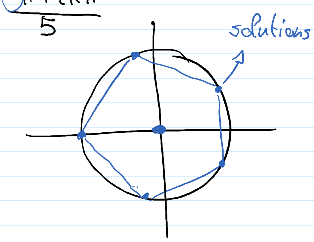
$\rho^7 e^{i\theta} = -2\rho^2 e^{i(2\theta \pm \pi)}$

$\begin{cases} |I| = |II| \\ \angle I = \angle II + 2k\pi \end{cases} \Rightarrow \begin{cases} \rho^7 = 2\rho^2 \\ \theta = 2\theta + \pi + 2k\pi \end{cases}$

$\theta = -\pi + 2k\pi = \pi + 2k\pi$

$\rho = 0 \lor \rho^5 = 2$

$z = 0 \quad \rho = \sqrt[5]{2} \quad \theta = \frac{-\pi + 2k\pi}{5}$




14) $|z+2| = \bar{z}^2 - 1$

$z = x + iy$

$|(x+2) + iy| = (x - iy)^2 - 1$

$\sqrt{(x+2)^2 + y^2} = x^2 - y^2 - 2ixy - 1$

$$\begin{cases} \sqrt{x^2 + 4x + 4 + y^2} = x^2 - y^2 - 1 \\ 2xy = 0 \end{cases} \quad x = 0 \ \wedge \ y = 0$$

I) $x = 0$ : $|y| = -y^2 - 1$ $\quad y^2 - |y| + 1 = 0$

$y > 0$ : $y^2 - y + 1 = 0$ $\quad \Delta = 1 - 4 < 0$

$y < 0$ : $y^2 + y + 1 = 0$ $\quad \Delta < 0$

II) $y = 0$ : $|x+2| = x^2 - 1$ $\quad x^2 - \underbrace{|x+2|}_{\geq 0} - 1 = 0$

$x \geq -2$ : $x^2 - x - 3 = 0$

$$x_{1,2} = \frac{1 \pm \sqrt{1+12}}{2} = \frac{1 \pm \sqrt{13}}{2}$$

$\frac{1+\sqrt{13}}{2} > -2$

$\frac{1-\sqrt{13}}{2}$ 4 3

$\frac{1-4}{2} = -1{,}5 > -2$

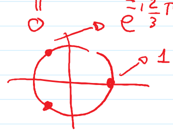
$$z = \frac{1 \pm \sqrt{13}}{2}$$

$x < -2$ $\quad x^2 + x + 1 = 0$ $\quad \Delta < 0$

$(|z - 6i| - |z + 4i|)(z^3 - 1) = 0$

$\overset{\parallel}{0}$ $\quad \overset{\parallel}{0}$ $\quad e^{\pm i \frac{2}{3}\pi}$

II) $z^3 = 1$




I) $|z - 6i| = |z + 4i|$

$|x + i(y-6)| = |x + i(y+4)|$

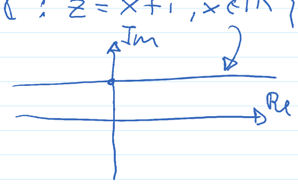
$\underbrace{\sqrt{x^2 + (y-6)^2}}_{\geq 0} = \underbrace{\sqrt{x^2 + (y+4)^2}}_{\geq 0}$

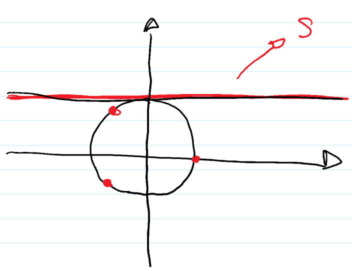
$\cancel{x^2} + (y-6)^2 = \cancel{x^2} + (y+4)^2$

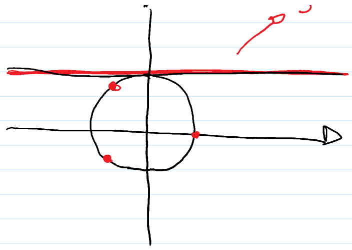
$\cancel{y^2} - 12y + 36 = \cancel{y^2} + 8y + 16$

$20y = 20 \quad \rightarrow \quad y = 1$

$\{ z \in \mathbb{C} : z = x + i, \ x \in \mathbb{R} \}$





3) $\begin{cases} \overline{\omega} z + 1 - i\sqrt{3} = 0 \\ z^3 |z|^2 - \overline{z}^2 \omega = 0 \end{cases}$ $\quad \overline{z} \cdot \overline{z} \omega$

$\rightarrow \overline{\omega} z = -1 + i\sqrt{3} \rightarrow \overline{\overline{\omega} z} = \omega \overline{z} = -1 - i\sqrt{3} = \overline{(-1 + i\sqrt{3})}$

$z^3 |z|^2 + (1 + i\sqrt{3}) \overline{z} = 0$

$\hookrightarrow |z|^2 = z \cdot \overline{z}$

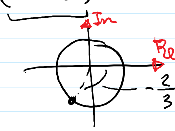
$\rightarrow \overline{z} (z^4 + 1 + i\sqrt{3}) = 0$

$\overline{z} = 0 \rightarrow z = 0$

$z^4 = -1 - i\sqrt{3} \rightarrow \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) 2$

$\sqrt{3+1} = 2$ $\quad \cos\theta = -\frac{1}{2}$, $\sin\theta = -\frac{\sqrt{3}}{2}$ $\rightarrow \theta = \frac{4}{3}\pi$ $\quad \rho = 2$

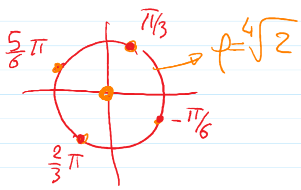
$z = x + iy \overset{!}{=} \rho e^{i\theta} \overset{!}{=} \rho(\cos\theta + i \sin\theta)$




$-\frac{2}{3}\pi = \frac{4}{3}\pi$

$\rho^z_{1234} = \sqrt[4]{2}$

$\alpha^z_{1234} = \dfrac{\frac{4}{3}\pi + 2k\pi}{4} = \frac{\pi}{3} + \frac{k\pi}{2}$




$\Longrightarrow \overline{\omega} = \dfrac{-1 + i\sqrt{3}}{z} \rightarrow \omega = \dfrac{-1 - i\sqrt{3}}{\overline{z}} = \dfrac{2 e^{i\frac{4}{3}\pi}}{\rho^z e^{-i\alpha^z}} = \dfrac{2}{\sqrt[4]{2}} e^{i\left(\frac{4}{3}\pi - \alpha^z\right)}$

$\hookrightarrow z \neq 0$

$z = 0$ : $1 - i\sqrt{3} = 0$ IMPOSSIBLE

$\Rightarrow \nexists \omega : z = 0 \rightarrow$ $z = 0$ is not a solution of the system

$\{(z, \omega) \in \mathbb{C}\}$

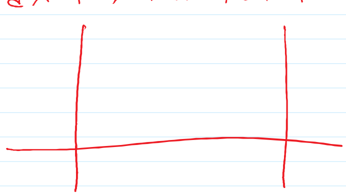
$S: \{(z, \omega) \in \mathbb{C}$ s.t. $z \in \{\sqrt[4]{2} e^{i\pi/3}, \sqrt[4]{2} e^{i\frac{5}{6}\pi}, \sqrt[4]{2} e^{i\frac{4}{3}\pi}, \sqrt[4]{2} e^{-i\frac{\pi}{6}}\}$

$\omega = \sqrt[4]{8}\, e^{i\left(\frac{4}{3}\pi - \angle z\right)} \}$

$\{(z, \omega) \in \{(\dots,\dots), (\dots,\dots), (\dots,\dots), (\dots,\dots)\}\}$

Ruffini's rule / formula

$ax^4 + bx^3 + cx^2 + dx + e = 0$



7) find $\lambda$ | $P(z)=z^3+2iz^2-4iz+\lambda$ has root $z=-2i$ + find all the roots

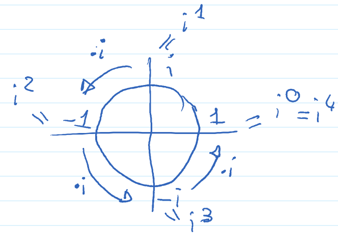
$P(-2i)=0$

$(-2i)^3+2i(-2i)^2-4i(-2i)+\lambda=0$

$-8(-i)+2i(+4(-1))+8(-1)+\lambda=0$

$\cancel{8i}\quad \cancel{-8i}-8+\lambda=0$

$\lambda=8\in\mathbb{R}$





$P(z)=z^3+2iz^2-4iz+8$

| | 1 | 2i | -4i | 8 |
|---|---|---|---|---|
| | | + | + | |
| -2i | | -2i | 0 | -8 |
| | 1 | 0 | -4i | 0 |

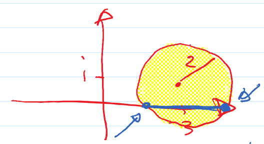
$P(z)=(z+2i)(z^2-4i)$

$z^2=4i=4e^{i\pi/2}$

$z=-2i \qquad z=2e^{i\frac{\pi}{4}},\ 2e^{i\frac{5}{4}\pi}$

10) $\begin{cases}|z-(3+i)|\leq 2\\ Re(z^2+7i)-(Re\,z)^2=0\end{cases}$

$z=x+iy$





I) $Re(z^2+7i)=(Re\,z)^2$

$\begin{cases}Re(x^2-y^2+2ixy+7i)=x^2-y^2\\ Re(x+iy)=x\end{cases} \Rightarrow \cancel{x^2}-y^2=\cancel{x^2}$

$y=0$

$\{z=x,\ x\in\mathbb{R}\}$

II) $\begin{cases}|x+iy-3-i|\leq 2\\ y=0\end{cases}$

$|(x-3)-i|\leq 2 \qquad \sqrt{\underbrace{(x-3)^2+1}_{x^2-6x+10}}\leq 2$

D: $x^2+6x+10>0$

$\frac{\Delta}{2}=9-10<0$



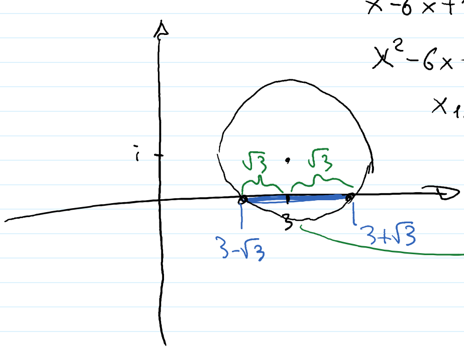
D: $x\in\mathbb{R}$

$x^2-6x+10\leq 4$

$x^2-6x+6\leq 0$

$x_{12}=3\pm\sqrt{9-6}=3\pm\sqrt{3}$

$x\in[3-\sqrt{3},\ 3+\sqrt{3}]$





13) $|z|^2+i\bar{z}+iz^3-i\bar{z}=0$

$z=\rho e^{i\theta}$

$\rho^2+\rho e^{i(\theta+\frac{\pi}{2})}+\rho^3e^{i(3\theta+\frac{\pi}{2})}+\rho e^{i(-\theta-\frac{\pi}{2})}=0$

$$\rho^2+\rho e^{i\left(\theta+\frac{\pi}{2}\right)}+\rho^3 e^{i\left(3\theta+\frac{\pi}{2}\right)}+\rho e^{i\left(-\theta-\frac{\pi}{2}\right)}=0$$

$$\rho\left(\rho+\underline{e^{i\left(\theta+\frac{\pi}{2}\right)}}+\rho^2\underline{e^{i\left(3\theta+\frac{\pi}{2}\right)}}+\underline{e^{-i\left(\theta+\frac{\pi}{2}\right)}}\right)=0$$

$$\rho=0 \longrightarrow z=0$$

$$\vee\quad \begin{cases} 3\theta+\frac{\pi}{2}=k\pi \\ \cancel{\theta}+\cancel{\frac{\pi}{2}}\cancel{-\theta}\cancel{-\frac{\pi}{2}}=k\pi \end{cases} \qquad \theta=-\frac{\pi}{6}+\frac{k\pi}{3}=\underbrace{\pm\frac{\pi}{6},\ \pm\frac{\pi}{2},\ \pm\frac{5}{6}\pi}_{e^{i\left(3\theta+\frac{\pi}{2}\right)}=\pm 1}$$

angles must be multiples of $\pi$
→ the equation has real coefficients
otherwise cannot be equal to 0

| | | | |
|---|---|---|---|
| A) $\theta=\frac{\pi}{6}$ | $-\rho^2+\rho+2\left(\mathrm{Re}\, e^{i\frac{2}{3}\pi}\right)=0$ | $-\rho^2+\rho-1=0$ | $\Delta<0$ |
| B) $\theta=-\frac{\pi}{6}$ | $\rho^2+\rho+2\left(\mathrm{Re}\, e^{i\frac{\pi}{3}}\right)=0$ | $\rho^2+\rho+1=0$ | $\Delta<0$ |
| C) $\theta=\frac{\pi}{2}$ | $\rho^2+\rho+2\left(\mathrm{Re}\, e^{i\pi}\right)=0$ | $\rho^2+\rho-2=0$ | $\cancel{\rho=-2} \vee \rho=1$ |
| D) $\theta=-\frac{\pi}{2}$ | $-\rho^2+\rho+2\left(\mathrm{Re}\, e^{i0}\right)=0$ | $-\rho^2+\rho+2=0$ | $\rho=2 \vee \cancel{\rho=-1}$ |
| E) $\theta=\frac{5}{6}\pi$ | $-\rho^2+\rho+2\left(\mathrm{Re}\, e^{i\frac{4}{3}\pi}\right)=0$ | $-\rho^2+\rho-1=0$ | $\Delta<0$ |
| F) $\theta=-\frac{5}{6}\pi$ | $\rho^2+\rho+2\left(\mathrm{Re}\, e^{-i\frac{\pi}{3}}\right)=0$ | $\rho^2+\rho+1=0$ | $\Delta<0$ |

$$\Longrightarrow S:\{0,\ i,\ -2i\}$$

# EXERCISES

| | |
|---|---|
| 6) $\sqrt[3]{1-i+\sqrt{2}i}$ | $\left\{\sqrt[3]{2},\ \sqrt[3]{2}e^{\pm i\frac{2}{3}\pi},\ \sqrt[3]{2}i,\ \sqrt[3]{2}e^{i\frac{7}{6}\pi},\ \sqrt[3]{2}e^{i\frac{11}{6}\pi}\right\}$ |
| 7) $\sqrt[3]{\frac{2}{\sqrt{2}}\left(\left\lvert\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right\rvert-ie^{2\pi i}\right)}$ | $\left\{\sqrt[3]{2}e^{-i\frac{\pi}{12}},\ \sqrt[3]{2}e^{i\frac{7}{12}\pi},\ \sqrt[3]{2}e^{-i\frac{9}{12}\pi}\right\}$ |
| 8) $\frac{\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)^5}{(1-i)^7}$ | $\left\{2^{-7/2}e^{-i\frac{\pi}{12}}\right\}$ |
| 9) $\frac{\left(-\frac{1}{2}+\frac{1}{2}i\right)^6}{(1-i\sqrt{3})^4}$ | $\left\{2^{-4}e^{-i\frac{\pi}{6}}\right\}$ |
| 10) $\frac{(2-2i)^4}{(-3\sqrt{3}-3i)^6}$ | $\left\{3^{-6}\right\}$ |
| 11) $\frac{\left(\frac{1}{3}+i\frac{\sqrt{3}}{3}\right)^5}{\left(\frac{1}{2}-\frac{1}{2}i\right)^4}$ | $\left\{2^{7}3^{-5}e^{i\frac{2}{3}\pi}\right\}$ |
| 15) $\lvert z\rvert z-2z+i=0$ | $\left\{i,\ -(1+\sqrt{2})i\right\}$ |
| 2) $\left(\lvert z-6i\rvert-\lvert z+2i\rvert\right)(z^3+1)=0$ | $\left\{x+2i \mid x\in\mathbb{R}\right\}\cup\left\{e^{i\frac{\pi}{4}},\ e^{i\frac{11}{12}\pi},\ e^{-\frac{5}{12}\pi}\right\}$ |
| 4) $\begin{cases}\bar{z}^2-w^2=-1\\ \bar{w}^2-z=0\end{cases}$ | $(z,w)\in\left\{\left(e^{i\frac{5}{3}\pi},e^{i\frac{\pi}{6}}\right),\left(e^{i\frac{\pi}{3}},e^{i\frac{5}{6}\pi}\right),\left(e^{i\frac{5}{3}\pi},e^{-i\frac{\pi}{6}}\right),\left(e^{i\frac{\pi}{3}},e^{-i\frac{5}{6}\pi}\right)\right\}$ |
| 5) $z^4-2i\sqrt{3}z^2-4=0$ | $\left\{\sqrt{2}e^{i\theta} \mid \theta\in\left\{\frac{\pi}{6},\frac{7}{6}\pi,\frac{\pi}{3},\frac{4}{3}\pi\right\}\right\}$ |

$\omega - z = 0$ $\quad$ $\left(e^{i\frac{5}{3}\pi}, e^{-i\frac{\pi}{6}}\right), \left(e^{i\frac{\pi}{3}}, e^{-i\frac{5}{6}\pi}\right)\}$

5) $z^4 - 2i\sqrt{3}z^2 - 4 = 0$ $\quad$ $\left\{\sqrt{2}e^{i\theta} \mid \theta \in \left\{\frac{\pi}{6}, \frac{7}{6}\pi, \frac{\pi}{3}, \frac{4}{3}\pi\right\}\right\}$

6) $i\,\mathrm{Re}(z) + z^2 = |z|^2 - 1$ $\quad$ $\left\{\pm\frac{\sqrt{2}}{2}i\right\}$

8) $\left(\mathrm{Re}\left(i\bar{z}(z-2)\right)\right)^2 = \left(\mathrm{Im}\left(z(\bar{z}-2i)\right)\right)^2$ $\quad$ $\{x \pm |x|i, \, x \in \mathbb{R}\}$

9) $z^4 - i\left|1 + i\sqrt{3}\right|z = 0$ $\quad$ $\left\{\sqrt[3]{2}e^{i\theta} \mid \theta \in \left\{\frac{\pi}{6}, \frac{5}{6}\pi, \frac{3}{2}\pi\right\}\right\} \cup \{0\}$

11) $(z+2\bar{z})^2 + |z-3|^2 - 10(\mathrm{Re}\,z)^2 = 0$ $\quad$ $\{3/2\}$

12) $\left(|z-3i|^2 + \mathrm{Re}(z+6\bar{z}) \cdot \mathrm{Im}(z-\bar{z})\,i - (7i+1)z\bar{z}\right) \in \mathbb{R}$ $\quad$ $\{x + xi, \, x \in \mathbb{R}\}$

14) $\frac{7}{e^{i\pi/2}} z^2\bar{z} + \frac{\mathrm{Im}\,z}{e^{i3\pi}} + 7i|z|^2\mathrm{Re}\,z = 0$ $\quad$ $\{x \mid x \in \mathbb{R}\} \cup \left\{x+iy \in \mathbb{C} \mid x^2+y^2 = \frac{1}{7}\right\}$

15) $z + 1 + i = -\frac{3i+1}{z+i-1}$ $\quad$ $\left\{\sqrt{\frac{3}{2}} - \left(\sqrt{\frac{3}{2}}+1\right)i, \, -\sqrt{\frac{3}{2}} - \left(\sqrt{\frac{3}{2}}-1\right)i\right\}$

16) $\left(\frac{z+1}{z-2}\right)^3 = -8$ $\quad$ $\{1, \, 2+\sqrt{3}i, \, 2-\sqrt{3}i\}$