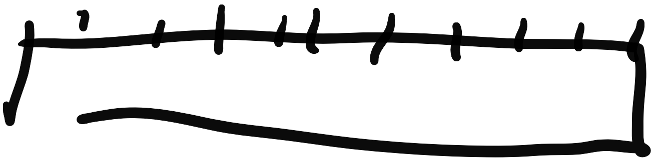
$$S = \{ s_1, s_2, \ldots, s_k \}$$



$k = 1$ $\quad 1_{arr} = 1$

$k = 2$ $\quad 2_{arr} = 2$

$k = 3$ $\quad 3_{arr} = 6$

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$k = 4$ $\quad 4_{arr} = 24$

$k = 5$ $\quad 5_{arr} = 120$

$\vdots$

.

.

arrangments of k elements are $k! = k(k-1)(k-2)\dots 2\cdot 1$

$\mathbb{N}$

FACTORIAL of k

Proof: by induction

i) Prove for $k=1$

yes

ii) If it is true for $k \Longrightarrow$ it is true for $k+1$

remembre

$k! = 1\cdot 2\cdot 3 \cdots (k-1)\cdot k$

$(k+1)! = 1\cdot 2\cdot 3 \cdots (k-1)k(k+1)$

$= k!\,(k+1)$

For each arrangem of k elements I can produce $k+1$ arrangements

Since there are $k!$ arrang of k elements

$$\Longrightarrow (k+1)\cdot k! = (k+1)!$$

__New sequence__

$k \longmapsto a_k := k!$

Set S with n elements

$k \leq n$ how many

subsets of k elements

I have?

$n = 3$ $\{a_1\ a_2\ a_3\}$

$k = 1$ $\{a_1\}\ \{a_2\}\ \{a_3\}$

$\{a_1, a_2\}\ \{a_1, a_3\}\ \{a_2, a_3\}$

Let us denote this number

$$\binom{n}{k} \qquad \binom{3}{1} = 3 \qquad \binom{3}{2} = 3$$

$$\binom{n}{1} = n \qquad \binom{n}{n-1} = n$$

$$\binom{n}{0} = 1$$

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots\cdots(n-k+1)}{k!}$$

1) choose the first one among n

2) choose the second one among $n-1$

3) choose the third one among $(n-2)$

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} =$$

$$\frac{n(n-1)\cdots(n-k+1)\overbrace{(n-k)(n-k-1)\cdots 2\cdot 1}^{(n-k)!}}{k!\,(n-k)(n-k-1)\cdots\cdots 1} =$$

$$= \frac{n!}{k!\,(n-k)!}$$

$$\binom{2}{0} = 1 \qquad \binom{2}{1} = 2 \qquad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \qquad \binom{3}{1} = 3 \qquad \binom{3}{2} = 3 \qquad \binom{3}{3} = 1$$

$$\binom{4}{0} = 1 \qquad \binom{4}{1} = 4 \qquad \binom{4}{2} = 6 \qquad \binom{4}{3} = 4 \qquad \binom{4}{4} = \underline{1}$$

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Theorem for every $a,b \in \mathbb{R}$ $\forall n \geq 1$

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{k}a^{n-k}b^k + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

Newton binomial

Proof

$$(a+b)^n = \overbrace{\underset{(1)}{(a+b)}\underset{(2)}{(a+b)}\cdots\cdots\underset{(n)}{(a+b)}}^{n \text{ times}} =$$

$$\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \dots + \binom{n}{k}a^{n-k}b^k + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

q.e.d.

Lemma Pascal triangle

$$\Rightarrow \boxed{\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}}$$

$(n-1)$ ... $k-1$ ... k.

(n)

$\Uparrow$ prove it.

Theorem: If a sequence (a_n) is increasing (decreasing) and then it has a limit

$$\lim_{n\to+\infty} a_n = \sup\{a_n\} \in \mathbb{R} \quad (\inf\{a_n\})$$

if $\{a_n\}$ is upper bounded (lower)

$$\lim_{n\to+\infty} a_n = +\infty \quad (-\infty)$$

if $\{a_n\}$ is not upper (lower) bounded.

Proof: Supp. $\{a_n\}$ is upper bounded

$\Rightarrow$ it has the supremum S

so that i) $S \geq a_n \quad \forall n \in \mathbb{N}$

$\beta = S - \varepsilon$ ii) $\forall \varepsilon > 0 \ \exists a_{\bar{n}}$

$$S - \varepsilon \leq a_{\bar{n}} \leq S$$

$$S - \varepsilon \leq a_{\bar{n}} \leq a_{\bar{n}+1} \cdots \leq a_n \leq S$$

$\forall n \geq \bar{n}$, i.e.

$\exists \bar{n}$ s.t. $\forall n \geq \bar{n}$

$$S - \varepsilon \leq a_n \leq S \leq S + \varepsilon$$

$$\boxed{S - \varepsilon \leq a_n \leq S + \varepsilon}$$

If $\{a_n\}$ is not upper bounded

i.e. $\forall K \in \mathbb{R} \quad \exists\, a_{\bar{n}}$ s.t.

$$a_n \geq \dots a_{\bar{n}+2} \geq a_{\bar{n}+1} \geq a_{\bar{n}} \geq K$$

$$\forall n \geq \bar{n} \qquad a_n \geq K,$$

that is,

$$\lim_{n \to \infty} a_n = +\infty$$

q.e.d.

(prove the decreasing version by exercise)

Consider the sequence

$$n \longmapsto a_n := \left(1 + \frac{1}{n}\right)^n$$

<u>Theorem</u> $\left(1+\frac{1}{n}\right)^n$ is increasing, and has a finite limit e

$$2 < e < 3$$

$$(1.5)^n$$

Proof

$$\underbrace{\left(\overset{a}{1}+\overset{b}{\frac{1}{n}}\right)^n} = \binom{n}{0}\cdot 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\left(\frac{1}{n}\right)^2$$

$$= \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k =$$

$$= \sum_{k=0}^{n} \frac{n(n-1)\cdots\cdots(n-k+1)}{k!\, n^k} =$$

$$\sum_{k=0}^{n} \frac{1}{k!}\frac{n}{n}\frac{(n-1)}{n}\cdot \quad \cdot\frac{(n-k+1)}{n} =$$

$$\sum \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot \quad \cdot\left(1-\frac{k-1}{n}\right)$$

Let us prove that $\left(1+\frac{1}{n}\right)^n$ is increasing

$$a_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} =$$

$$= \sum_{k=0}^{n+1} \frac{1}{k!} \left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) \cdot \ldots \cdot \left(1-\frac{k}{n+1}\right)$$

$$\geq \sum_{k=0}^{n+1} \frac{1}{k!} \left(1-\frac{1}{n}\right) \cdot \ldots \cdot \left(1-\frac{k}{n}\right)$$

$$\geq \sum_{k=0}^{n} \frac{1}{k!} \left(1-\frac{1}{n}\right) \cdot \ldots \cdot \left(1-\frac{k}{n}\right) =$$

$$= \left(1+\frac{1}{n}\right)^n = a_n$$

$$\Longrightarrow \quad a_{n+1} \geq a_n$$