Lesson 11 - 20/10/2022

Phase portraits of $\ddot{x} = -V'(x)$

1) Curves of the phase portrait are levels of the total energy $E(x,v) = \frac{1}{2}v^2 + V(x)$ since it is a FIRST INTEGRAL (conserved quantity) of the motion.

2) Curves of the phase portrait have vertical tangent only when they cross the x-axis (where $\dot{x} = v = 0$)

3) Curves of the phase portrait are symmetric with respect to the x-axis. $\frac{1}{2}v^2 + V(x) = c \iff v^2 = 2(c - V(x))$

$\iff v = \pm\sqrt{2(c-V(x))}$

4) Since $\frac{1}{2}v^2 + V(x) = c$ and $\frac{1}{2}v^2 \geq 0$ then $c - V(x) \geq 0 \iff V(x) \leq c \to$ Fixed $c \in \mathbb{R}$ for the corresponding orbits the inequality $V(x) \leq c$ holds $\to$ There are parts of the configuration space where orbits cannot "live".

5) $E(x,v) = \frac{1}{2}v^2 + V(x) = c$ $\quad c$ fixed.

Are the corresponding curves regular?

This holds iff $\begin{cases} \partial E/\partial x = V'(x) \\ \partial E/\partial v = v \end{cases}$ has max rank $(=1)$

This cond. doesn't hold when $\begin{cases} V'(x) = 0 \\ v = 0 \end{cases}$ exactly the EQUILIBRIA

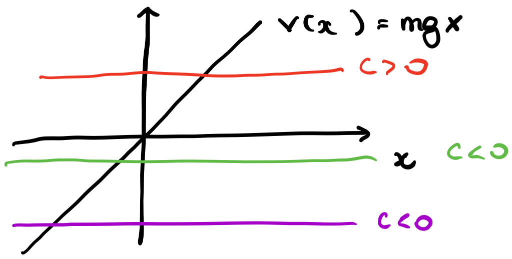
THEN:

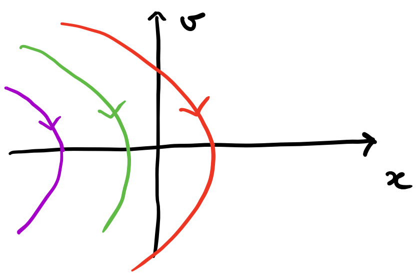
Curves of the phase portraits are regular outside equilibria. Unique singular points of $N_c = \{(x,v) \in \mathbb{R}^2 : E(x,v) = c\}$ (isolated points, auto-intersections, angular points, cuspids...) are equilibria.

EXAMPLES

- Gravitational force near earth

$m\ddot{x} = -mg \Rightarrow V(x) = mgx$



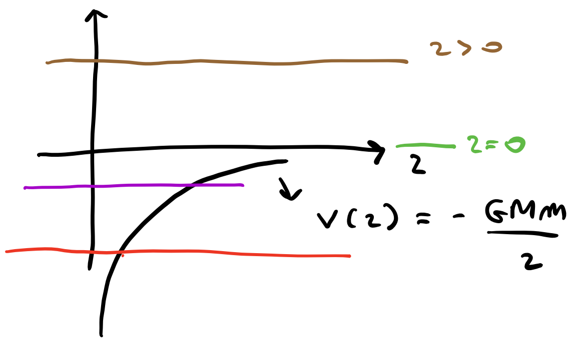


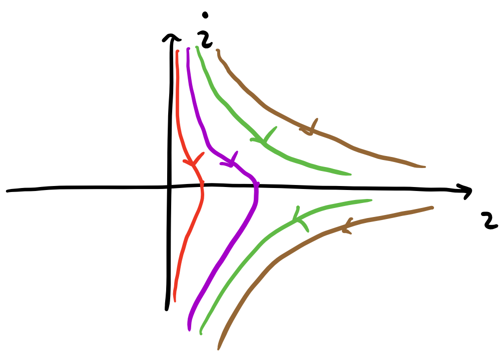
- Keplerian gravitation force

$$m\ddot{z} = -\frac{GMm}{z^2} \Rightarrow V(z) = -\frac{GMm}{z}$$

$\downarrow$

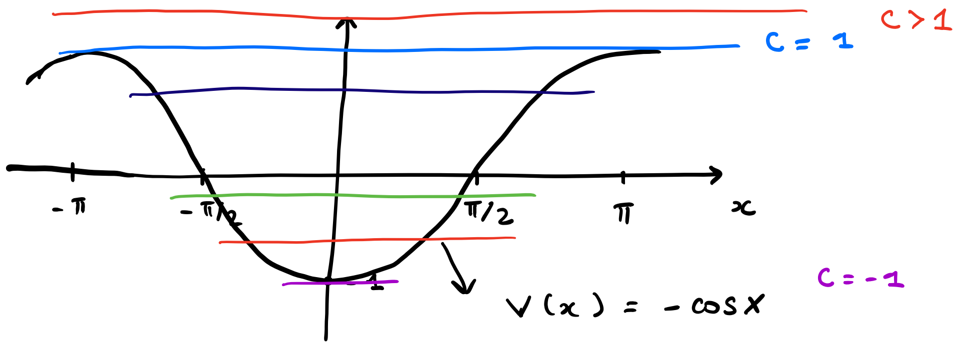
$z>0$

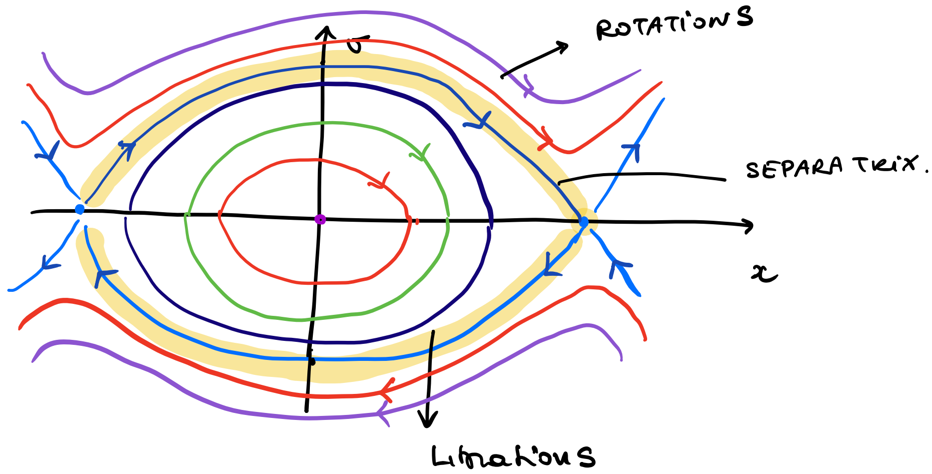




- Pendulum

$$\ddot{x} = -\sin x \Rightarrow V(x) = -\cos x$$





For $C = 1$, THE CORRESPONDING ORBITS ARE 3

By the periodicity of the $\cos x$, the "best" phase-space for the pendulum is the cilinder.

EXERCISE

Solve the Cauchy problem

$$\begin{cases} \ddot{x} = \underbrace{4x^3 + 4x}_{= f(x) = -V'(x)} \\ x(0) = 0 \\ \dot{x}(0) = \sqrt{2} \end{cases}$$

SOL

We can use the first integral $E(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + V(x)$

Since

$$4x^3 + 4x = -V'(x) \Rightarrow V(x) = -\int_0^x (4s^3 + 4s)\,ds =$$

$$= -\left[s^4 + 2s^2 \right]_0^x = -x^4 - 2x^2$$

$$E(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - x^4 - 2x^2 = E(0, \sqrt{2}) = 1$$

$$\Rightarrow \quad \frac{1}{2}\dot{x}^2 - x^4 - 2x^2 = 1$$

$$\Rightarrow \quad \dot{x}^2 = 2\,(x^4 + 2x^2 + 1)$$

$$\Rightarrow \quad \dot{x} = \underbrace{\operatorname{sgn}(\dot{x}(0))}_{>0 \text{ in our case}} \sqrt{2}\,\sqrt{x^4 + 2x^2 + 1}$$

$\downarrow$

= until the first instant where $\dot{x} = 0$

$$\dot{x} = \sqrt{2}\,\underbrace{\sqrt{x^4 + 2x^2 + 1}}_{\sqrt{(x^2+1)^2} \,=\, (x^2+1)}.$$

we integrate by separation of variables.

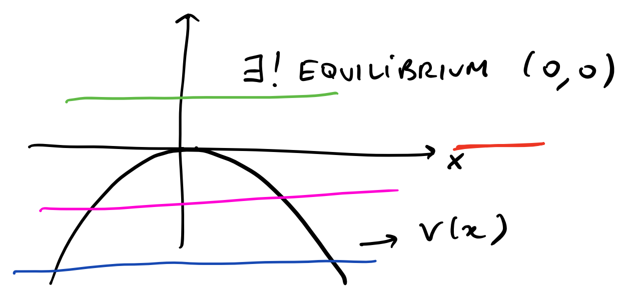
$$\frac{dx}{dt} = \sqrt{2}\,(x^2 + 1) \quad \Rightarrow \quad \int_0^t \sqrt{2}\, ds = \int_{0 = x_0}^{x} \frac{1}{y^2 + 1}\, dy$$

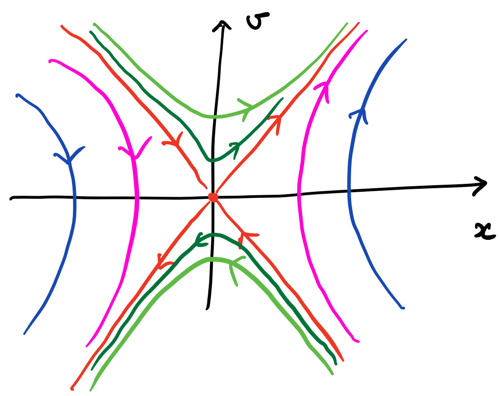
$$\Rightarrow \quad \sqrt{2}\,(t - 0) = \arctan y \Big|_0^x = \arctan x$$

$$\Rightarrow \quad \sqrt{2}\, t = \arctan x \quad \Rightarrow \quad \boxed{x(t; 0, \sqrt{2}) = \tan(\sqrt{2}\, t)}$$

Phase portrait

$V(x) = -x^4 - 2x^2$





Clearly $(0,0)$ is unstable (it's a local max of the potential !!). Other check $\rightarrow$ with Lyapunov method.

$$\begin{cases} \dot{x} = v \\ \dot{v} = 4x^3 + 4x \end{cases} \qquad JX(x,v) = \begin{pmatrix} 0 & 1 \\ 12x^2+4 & 0 \end{pmatrix}$$

$$JX(0,0) = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \quad \text{with } \lambda_{1,2} = \pm 2 \rightarrow \underline{\text{unstability}}$$

$\boxed{\text{EXERCISE}}$

Describe qualitatively the motion of a point subjected to the conservative force with potential energy:

$$V(x) = x^3/3 - x^2/2 - x$$

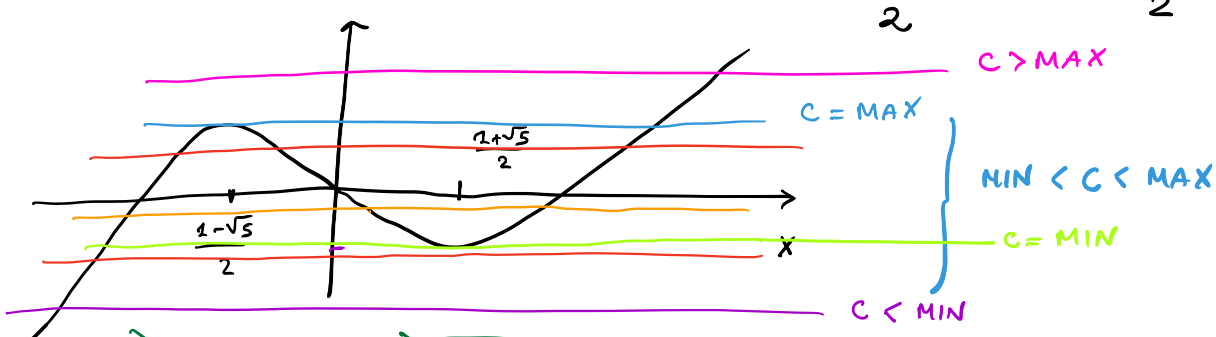
$\boxed{\text{SOL}}$

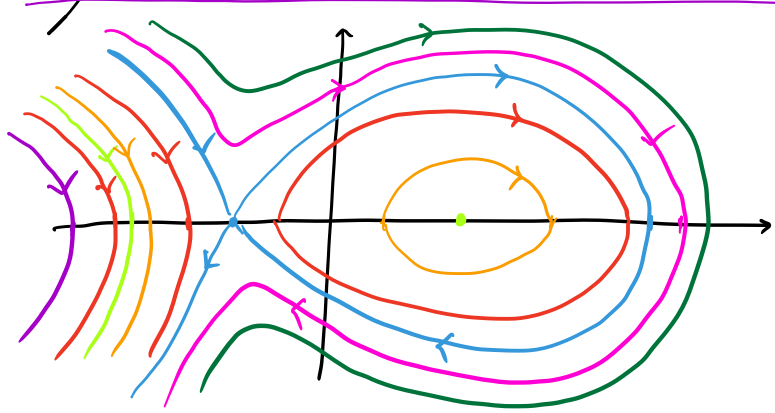
Draw the graph of $V(x)$.

$$V(x) = x\left(x^2/3 - x/2 - 1\right)$$

$$\lim_{x \to \pm\infty} V(x) = \pm\infty$$

$$V'(x) = x^2 - x - 1 \rightarrow x_{1,2} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$





Q. Values of energy corresponding to periodic motions ??

A. $MIN \leq c \leq MAX$

$MIN \downarrow$ fixed point (stable)

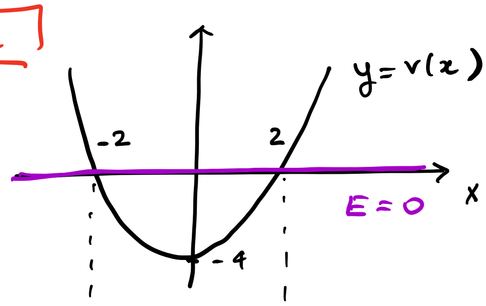
$MAX \downarrow$ fixed point (unstable)

$\left(\mathrm{Fix} \subseteq \mathrm{Per} \right)$

EXERCISE

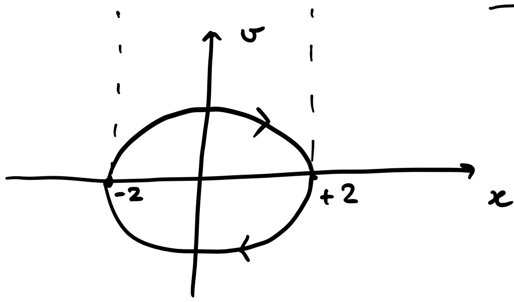
Calculate the period of motion given by
$$\begin{cases} \ddot{x} = -v'(x) \\ x_0 = 2 \\ \dot{x}_0 = 0 \end{cases}$$
where $v(x) = x^2 - 4$

SOL



→ The phase-space is foliated by invariant curve.

$$E(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + v(x) = \underline{\frac{1}{2}\dot{x}^2 + x^2 - 4 = E(2,0) = 0}$$



$$\frac{1}{2}\dot{x}^2 + x^2 - 4 = 0 \quad \Rightarrow \quad \dot{x}^2 = 2(4 - x^2)$$

$$\Leftrightarrow \quad \underset{\frac{dx}{dt}}{\underset{\parallel}{\dot{x}}} = \mathrm{sgn}(\dot{x}_0)\sqrt{2(4-x^2)}$$

$$\Rightarrow \quad T = \text{period} = 2\int_{-2}^{2} \frac{dy}{\sqrt{2(4-y^2)}} = \frac{2}{\sqrt{2}}\int_{-2}^{2} \frac{dy}{\sqrt{4-y^2}} =$$

$$= \sqrt{2}\,\arcsin\left(y/2\right)\Big|_{-2}^{2} = \sqrt{2}\left(\pi/2 + \pi/2\right) = \sqrt{2}\,\pi$$

—x—x—x—