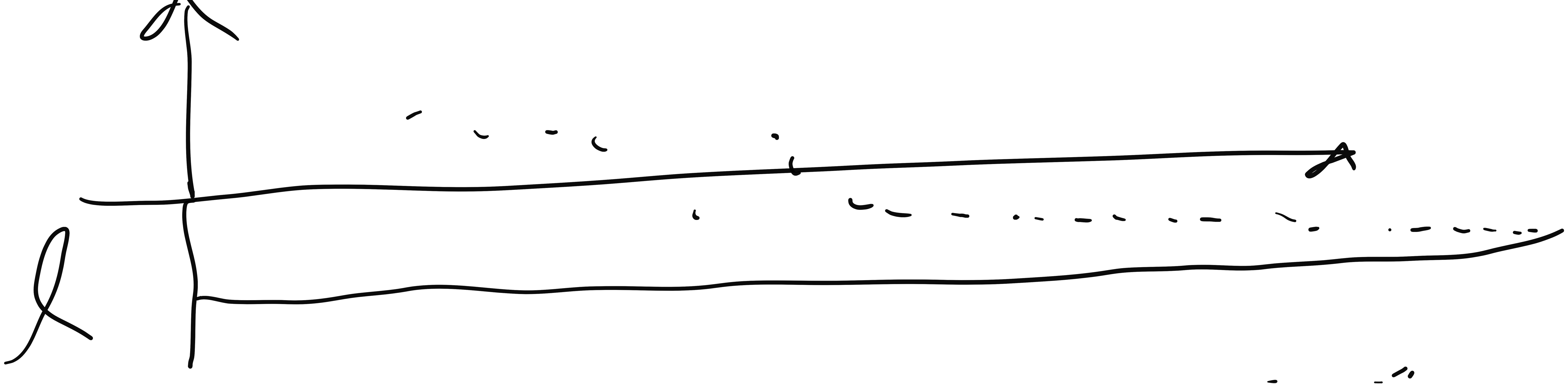
(a_n) a sequence

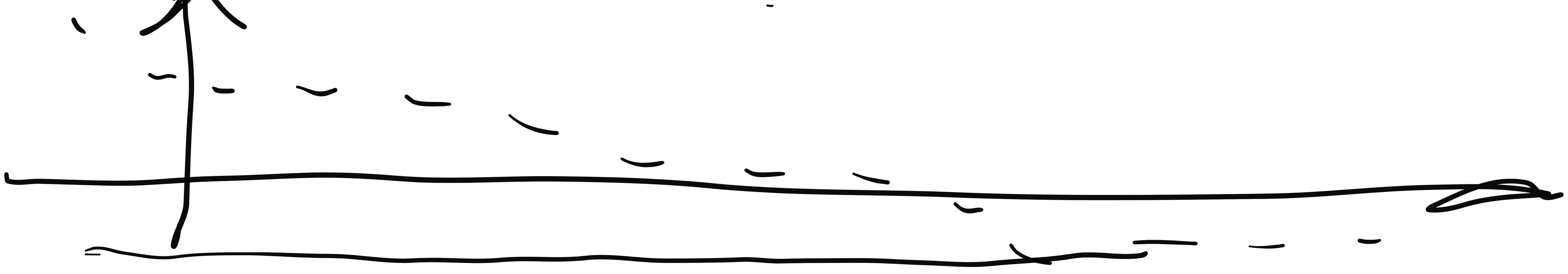
Definitions:

$\lim_{n\to\infty} a_n = l \in \mathbb{R}$ if $\forall \varepsilon > 0$
$\exists N : |a_n - l| < \varepsilon \quad \forall n \geq N$

$\lim_{n\to\infty} a_n = +\infty$ if $\forall K \in \mathbb{R}$
$\exists N \quad a_n > K \quad \forall n \geq N$

$\lim_{n\to\infty} a_n = -\infty$ if $\forall K \in \mathbb{R}$
$\exists N \quad a_n < K \quad \forall n \in N$





"Permanence of sign"

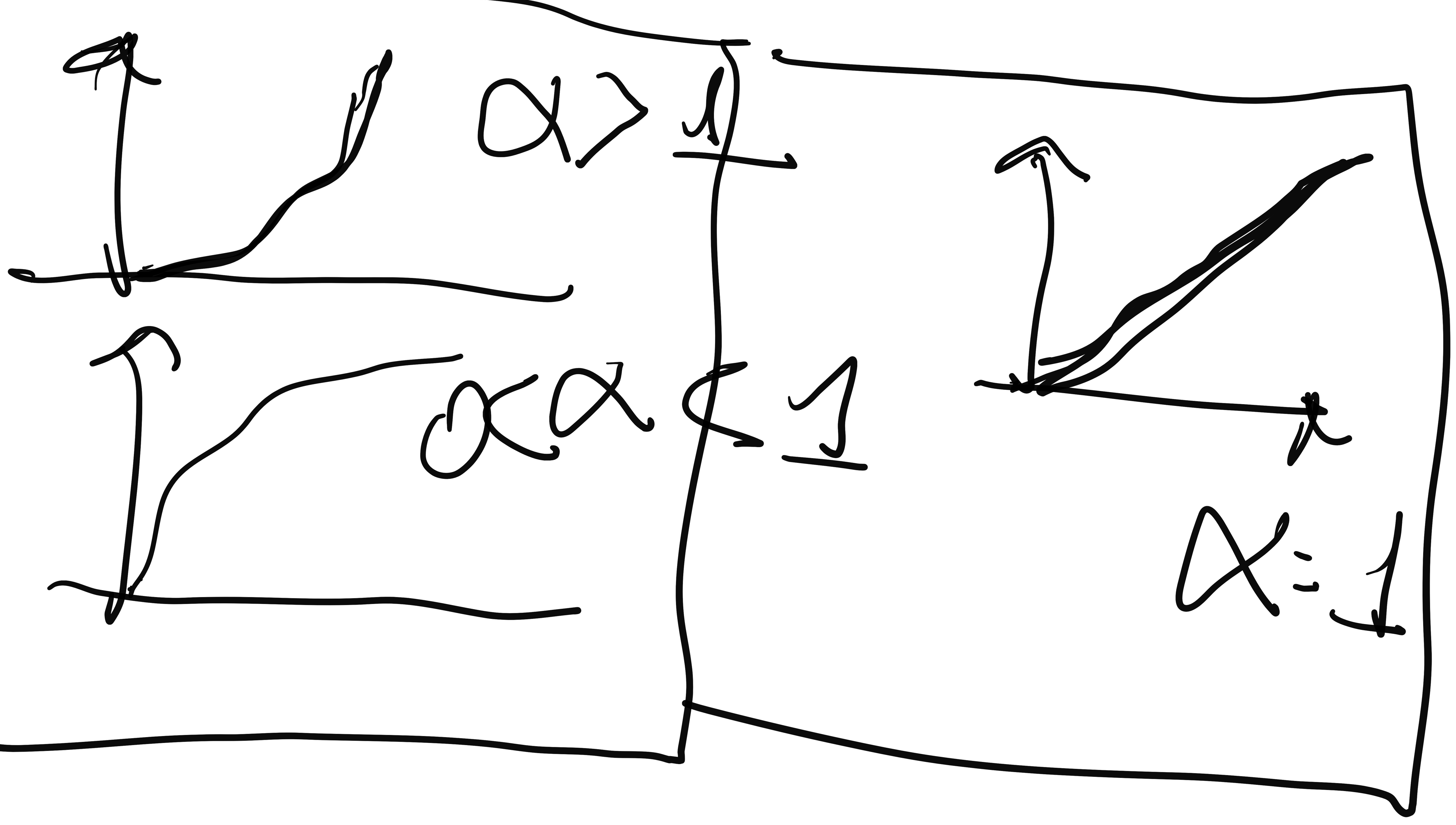
Theorem: if $\lim a_n = l > 0$ $\left(\begin{smallmatrix}+\infty \\ < \\ -\infty\end{smallmatrix}\right)$

then $a_n > 0$ $(<)$ definitely

$\Updownarrow$

$\forall n \geq \boxed{N}$

$\alpha > 0 \quad | \quad a_n = n^\alpha$



$\alpha > 1$

$0 < \alpha < 1$

$\alpha = 1$

$\lim n^\alpha = +\infty$

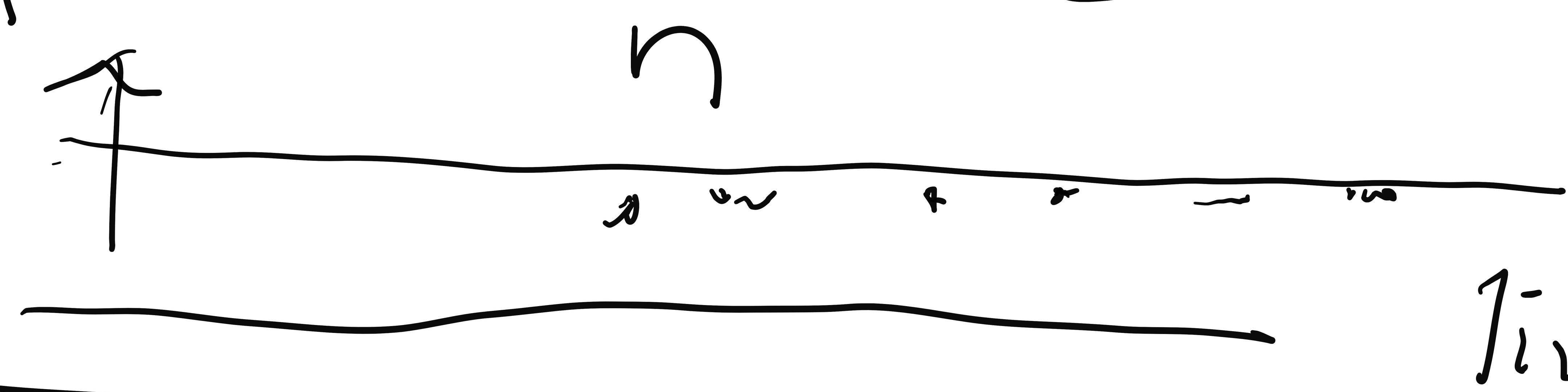
Indeed

$\forall K > 0 \quad n^\alpha \geq K$

$\Updownarrow$

$n \geq K^{\frac{1}{\alpha}} \triangleq N$

($a_n = 1 - \frac{1}{n}$ is increasing



but $\lim a_n = 1$)

$a>1$ $\quad a^n \qquad \lim a^n = +\infty$

$K \qquad a^n \geq K \iff n \geq \log_a K$

$a=1 \qquad \lim 1^n = 1$

$0<a<1 \qquad \lim a^n = 0 \quad \forall \varepsilon > 0$

$$-\varepsilon \leq a^n \leq \varepsilon$$

$$n \geq \log_a \varepsilon = N$$

Definition (subsequence):

Given a sequence $a_0, a_1, a_2, a_3, a_4, \dots$

Let n_k an strictly increasing sequence

$n_k \in \mathbb{N} \ \forall k$

$n_0 < n_1 < n_2 < n_3 < \dots$

the sequence $(a_{n_k})_{k \in \mathbb{N}}$ is

a SUBSEQUENCE of (a_n)

$a_0, a_1, a_2, a_3, a_4, a_5$

$a_0 \quad a_2 \quad a_4 \quad a_6 \cdots$ subseq.

$a_0 \quad a_3 \quad a_6 \quad a_9 \cdots$

$a_6 \quad a_0 \quad a_{12} \quad a_5 \quad a_{18} \quad a_{13} \cdots$

not subsequence.

Theorem: $(a_n)_{n \in \mathbb{N}}$ be a sequence having a limit

$$\lim_{n \to \infty} a_n = l \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$$

Then $\forall$ any subsequence

$$(a_{n_k})_{k \in \mathbb{N}} \quad \lim_{k \to \infty} a_{n_k} = l$$

Proof case $l \in \mathbb{R}$ We want to prove.

$\forall \varepsilon > 0 \ \exists N$ s.t $|a_{n_k} - l| \leq \varepsilon$ $\forall k \geq N$.

Our hipothesis $\exists \tilde{N}$ s.th.

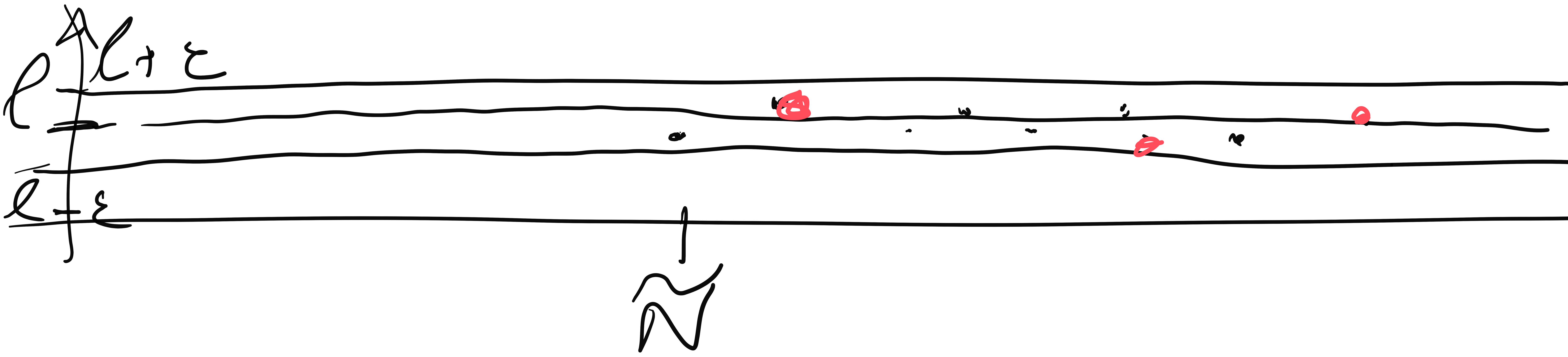
$(*)$ $|a_n - \ell| \leq \varepsilon \quad \forall\, n > \tilde{N}$

there will exist a number N

$n_N \geq \tilde{N}$ from $(*)$

$\boxed{|a_{n_k} - \ell| \leq \varepsilon} \quad \forall\, k \geq N$

(indeed $n_k > n_N \geq \tilde{N}$)



Corollary: (a_n) is a sequence and (a_{n_k}) is a subs. $(a_{n_{k'}})$ is a subs.

$$\lim_{k\to\infty} a_{n_k} = \ell_1$$

$$\lim_{k'\to\infty} a_{n_{k'}} = \ell_2$$

if $\ell_1 \neq \ell_2 \implies (a_n)$ doesn't have limit.

Proof. From Th above we have

$$\underbrace{\lim a_n = l}_{A} \Longrightarrow \underbrace{\begin{matrix}\lim a_{n_k} = l \\ \lim a_{n_{k'}} = l\end{matrix}}_{B}$$

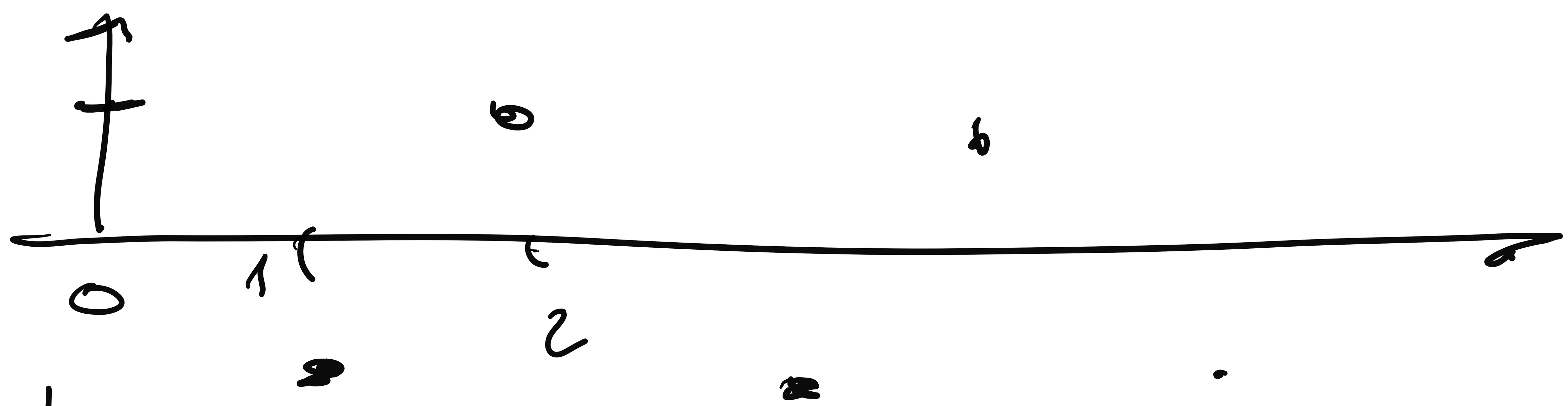
$$A \Longrightarrow B$$

$$\Updownarrow$$

$$\text{not } B \Longrightarrow \text{not } A$$

$$\left.\begin{matrix}\lim a_{n_k} = l_1 \\ \lim a_{n_{k'}} = l_2 \\ l_1 \neq l_2\end{matrix}\right\} \Longrightarrow (a_n) \text{ doesn't have a limit.}$$

$a_n = (-1)^n$



$n_k = 2k \qquad a_{n_k} = a_{2k} = (-1)^{2k} = 1$

$$\lim_{k \to \infty} a_{n_k} = 1$$

$n_{k'} = 2k' + 1 \qquad \lim_{k' \to \infty} a_{n_{k'}} = \lim (-1)^{2k'+1} = -1$

$$a_n = \sin\left(n\frac{\pi}{2}\right)$$

$$a_{n_k} = \sin\left((4k+1)\frac{\pi}{2}\right) =$$

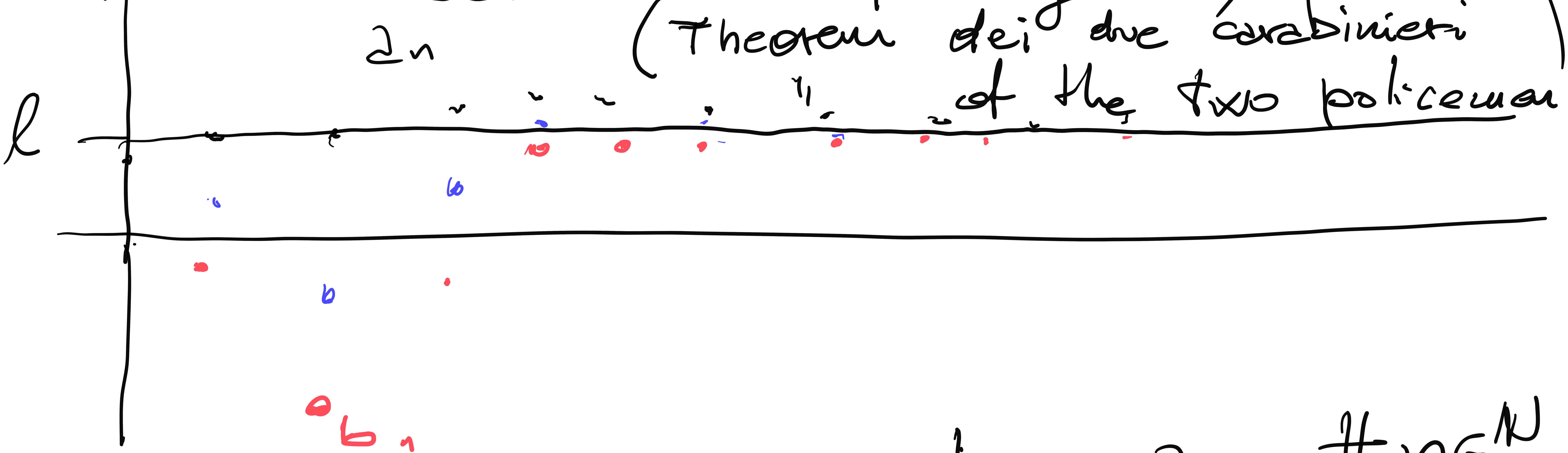
$$= \sin\left(2k\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right)$$

$$= 1$$

$$a_{n_{k'}} = \sin\left(4k' \cdot \frac{\pi}{2}\right) = \sin(2k'\pi) = 0$$

$$\left.\begin{array}{l}\lim a_{n_k} = 1 \\ \qquad\qquad \ne \\ \lim a_{n_{k'}} = 0\end{array}\right\} \Longrightarrow (a_n) \text{ has no limit.}$$

THEOREM (the squeezing th.)
(Theorem dei due carabinieri
of the two policemen



$(a_n)_{n\in\mathbb{N}}$ $\quad (b_n)_{n\in\mathbb{N}}$ $\quad b_n \le a_n \quad \forall n\in\mathbb{N}$ and $\lim a_n = \lim b_n = \ell$

(c_n) a sequence s.t. $\underline{b_n \le c_n \le a_n}$ $\quad \forall n\in\mathbb{N}$

$$\lim_{n\to\infty} c_n = l$$

Proof. $\forall \varepsilon > 0 \ \exists N_1, N_2$:

$$l - \varepsilon \leq a_n \leq l + \varepsilon \quad \forall n \geq N_1$$

$$l - \varepsilon \leq b_n \leq l + \varepsilon \quad \forall n \geq N_2$$

$N := \max\{N_1, N_2\}$. For $n \geq N$

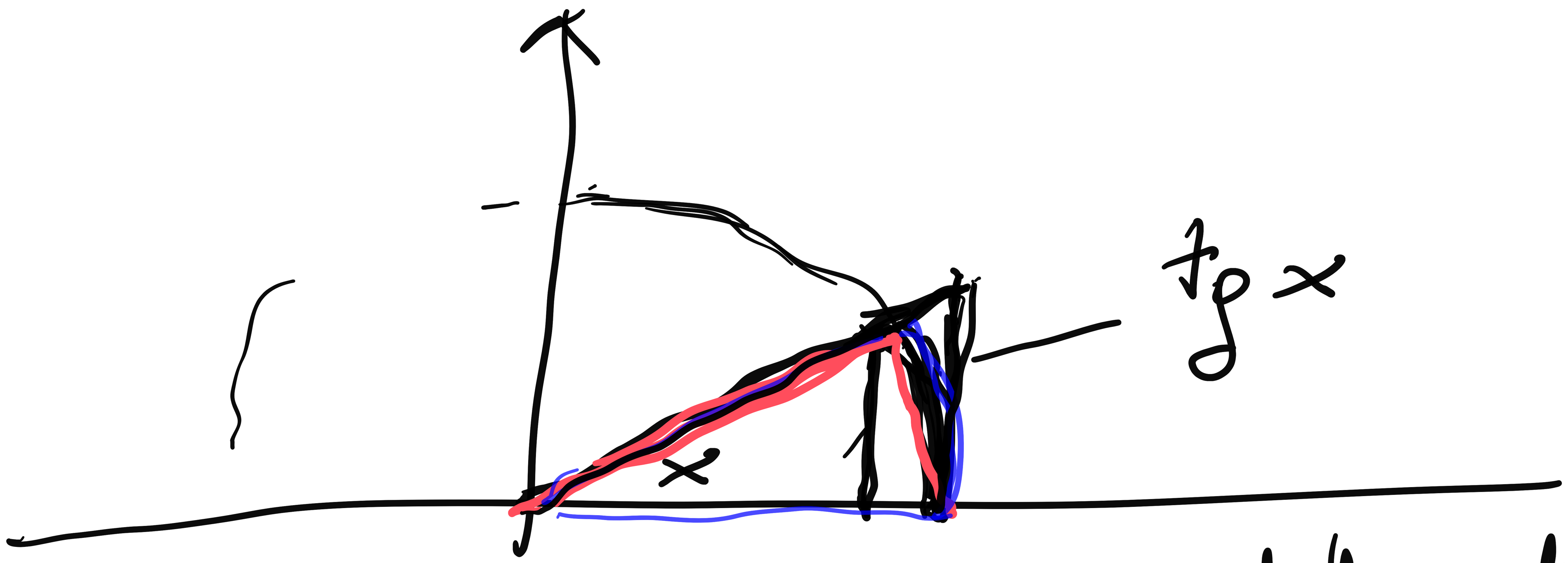
$$l - \varepsilon \leq b_n \leq c_n \leq a_n \leq l + \varepsilon$$

$$\Downarrow$$

$$l - \varepsilon \leq c_n \leq l + \varepsilon \quad \forall n \geq N$$

i.e. $\lim_{n\to\infty} c_n = l$

Exercise: $a_n = n \sin\left(\frac{1}{n}\right) = \dfrac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$, $\sin\left(\frac{1}{n}\right) \to 0$, $\frac{1}{n} \to 0$



$x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \setminus \{0\}$

$\boxed{x > 0}$

area of the sector

area of big triang

area of small triangle

$$\frac{\sin x}{2} \leq \frac{x}{2} \leq \frac{tg\, x}{2}$$

$$\sin x \leq x \leq \frac{\sin x}{\cos x}$$

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

$$\cos x \leq \frac{\sin x}{x} \leq 1$$

$x = \frac{1}{n}$

$$\underbrace{\cos\frac{1}{n}}_{b_n} \leq \underbrace{\frac{\sin\frac{1}{n}}{\frac{1}{n}}}_{c_n} \leq \underbrace{1}_{a_n}$$

exercise

$$\lim b_n = \lim \cos\frac{1}{n} = 1 \qquad \lim a_n = 1$$

$$\underset{\text{By 2 Pol. Th.}}{\Longrightarrow} \lim \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim n \sin\frac{1}{n} = 1$$

Theorem $\lim a_n = l_1 \in \mathbb{R}$ $\lim b_n = l_2 \in \mathbb{R}$,

then $\lim\limits_{n\to\infty} (a_n + b_n) = l_1 + l_2$

Proof $\forall \varepsilon > 0$ $\exists N_1, N_2$ s.t.

$(*)$ $\quad l_1 - \frac{\varepsilon}{2} \leq a_n \leq l_1 + \frac{\varepsilon}{2} \quad \forall n \geq N_1$

$(**)$ $\quad l_2 - \frac{\varepsilon}{2} \leq b_n \leq l_2 + \frac{\varepsilon}{2} \quad \forall n \geq N_2$

$N = \max\{N_1, N_2\}$. Adding $(*)$ and $(**)$

$$(l_1 + l_2) - \varepsilon \leq a_n + b_n \leq (l_1 + l_2) + \varepsilon$$

$$\Longrightarrow \lim a_n + b_n = l_1 + l_2$$
