Examples:

1) $\boxed{a_n = \frac{n+1}{n-1}}$ $\quad \forall n \in \mathbb{N} \setminus \{1\}$ $\quad \lim_{n \to +\infty} a_n = 1$

2) $a_n = \frac{n}{\sqrt{n} + 5}$ $\quad \forall n \in \mathbb{N}$

3) $\boxed{a_n = (-1)^n}$ $\quad \forall n \in \mathbb{N}$

4) $a_n = \sin(n\pi)$ $\quad \forall n \in \mathbb{N}$.

5) $\hat{a}_n = \frac{\sqrt{n} + 5}{n}$

Definition: $l \in \mathbb{R}$

$\lim_{n \to +\infty} a_n = l$ if

$\forall \varepsilon > 0 \quad \exists N \text{ s.t. } \forall n \geq N$

$$l - \varepsilon \leq a_n \leq l + \varepsilon$$

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5) $\lim_{n \to +\infty} \hat{a}_n = 0$

$\forall \varepsilon > 0 \quad \exists N \text{ s.t. } -\varepsilon < \hat{a}_n < \varepsilon$

$$-\varepsilon < \frac{\sqrt{n}+5}{n} < \varepsilon$$

Trivial, i.e. it is true for every $n \in \mathbb{N}$, $n \neq 0$

$\Updownarrow$

$$\sqrt{n}+5 < \varepsilon n$$

$$\sqrt{n} \leq \varepsilon n - 5$$

$\varepsilon n - 5 \geq 0$

$n \geq \frac{5}{\varepsilon}$

Alternatively:

$y = \sqrt{n}$

$y \leq \varepsilon y^2 - 5$

$$\begin{cases} n \geq \frac{5}{\varepsilon} \\ n \leq \varepsilon^2 n^2 - 10\varepsilon n + 25 \end{cases}$$

$$\begin{cases} n \geq \frac{5}{\varepsilon} \\ \varepsilon^2 n^2 - n(10\varepsilon + 1) + 25 \geq 0 \end{cases}$$

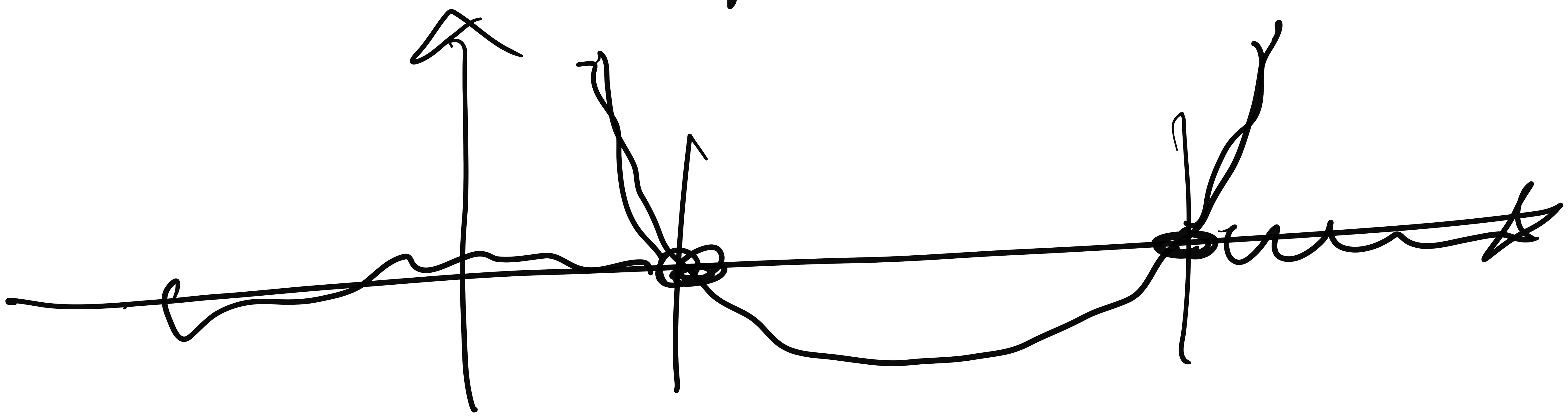
$$x_{1,2} = \frac{10\varepsilon + 1 \pm \sqrt{\Delta(\varepsilon)}}{2\varepsilon^2}$$

I) $\Delta(\varepsilon) = (10\varepsilon + 1)^2 - 100\varepsilon^2 < 0$

II) " " " ≥ 0

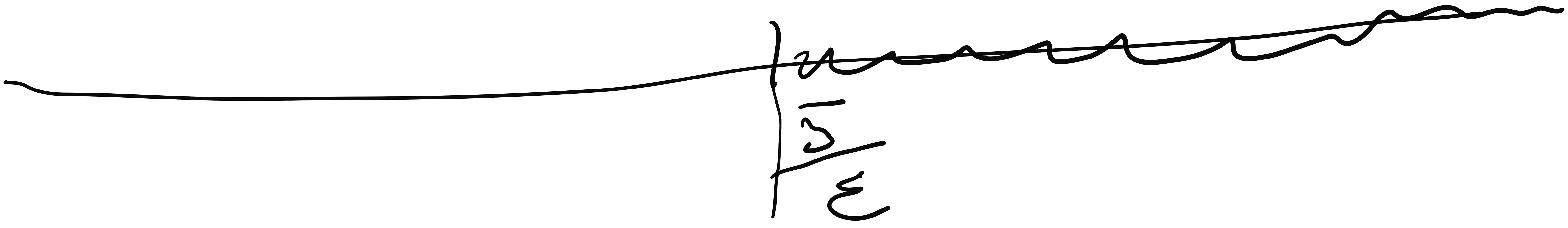
$$\begin{cases} ax^2+bx+c \geq 0 \\ a>0 \\ \boxed{\Delta < 0} \end{cases}$$

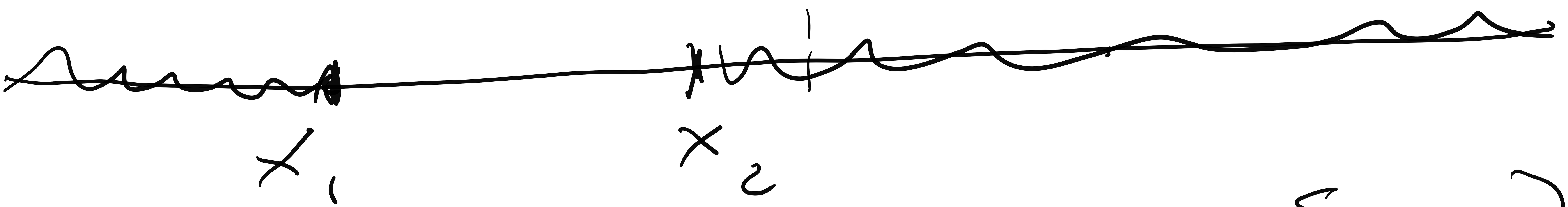
$\Delta = b^2 - 4ac$



I) $\begin{cases} n \geq \dfrac{5}{\varepsilon} \\ \mathbb{N} \end{cases}$ $\qquad n \geq \dfrac{5}{\varepsilon} = \underline{\underline{N}}$

II) $\begin{cases} n \geq \dfrac{5}{\varepsilon} \\ \Delta > 0 \ \ldots \quad x_1, x_2 \end{cases}$





III) ok $\Longleftrightarrow$ $n \geq \max\left\{\dfrac{5}{\varepsilon}, x_2\right\} = N$

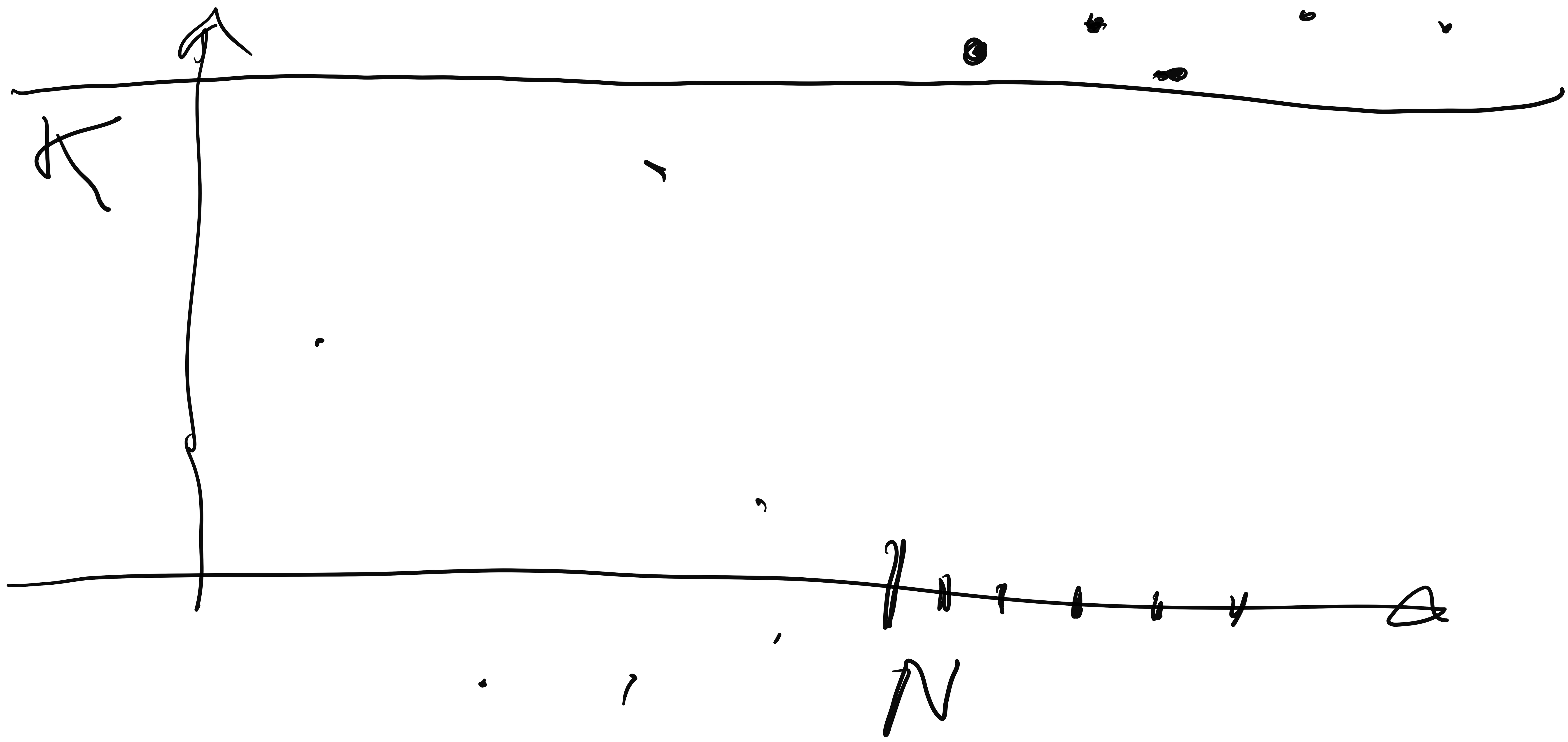
2) $a_n = \dfrac{n}{\sqrt{n}+5} \approx \dfrac{1}{a_n} \to 0$

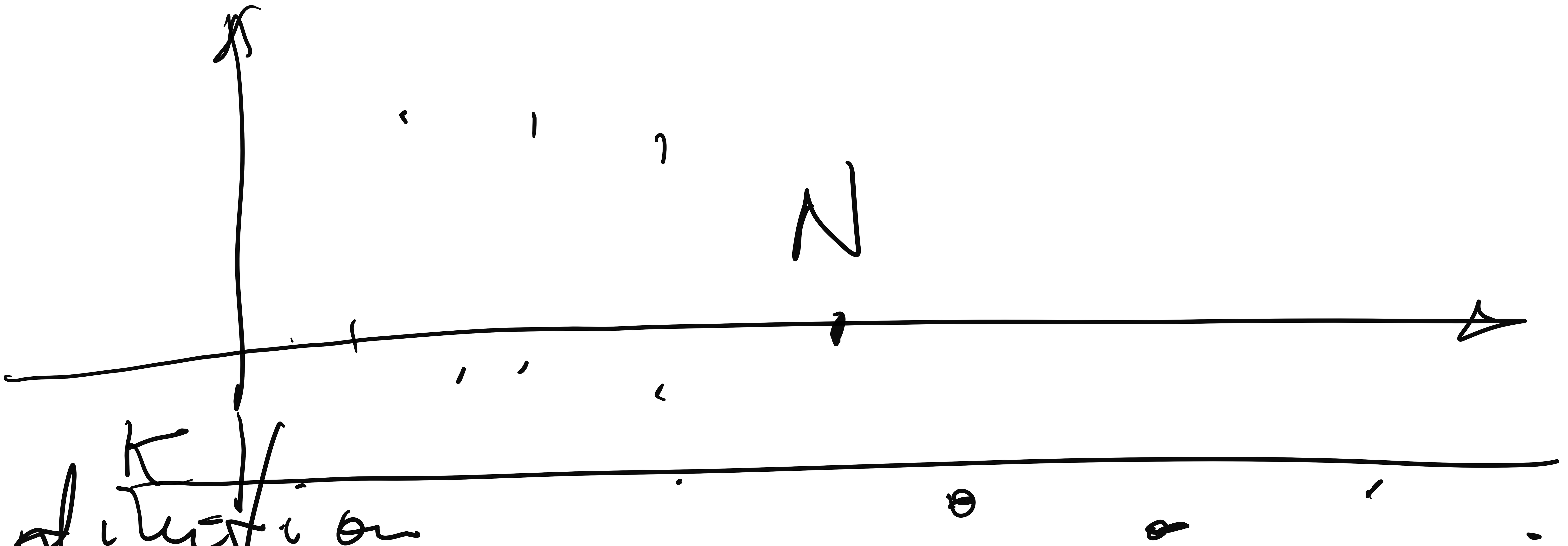
a_n is "larger and larger"

<u>Definition</u> We say that a_n tends to $+\infty$, and write

$\boxed{\lim_{n\to+\infty} a_n = +\infty}$ (or $a_n \to +\infty$)

if $\boxed{\forall K > 0}$ $\exists N$ s.t. $\forall n \geq N$ $a_n \geq K$.





Definition

$\lim_{n\to\infty} a_n = -\infty$ $\quad \forall K \in \mathbb{R}$

$\exists N \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $\underline{n \geq N}$

$a_n \leq K$

$$a_n = \frac{n \to +\infty}{\sqrt{n}+5 \to +\infty} \longrightarrow +\infty$$

$\forall K > 0$ $\quad \frac{n}{\sqrt{n}+5} > K \quad (*)$

$\exists N$ s.t. $(*)$ is o.k

$\forall n \geq N$

complete with calculations

3) $a_n = (-1)^n$

no limit

complete!

4) $a_n = \sin\left(\frac{\pi}{2} n\right)$

no limit

complete!

UNIQUENESS OF THE LIMIT

Theorem $\left.\begin{array}{l}\lim a_n = l_1 \in \mathbb{R} \\ \lim a_n = l_2 \in \mathbb{R}\end{array}\right\} \Rightarrow l_1 = l_2$

Proof:

By contradiction

suppose $l_1 \neq l_2$

choose $\varepsilon = \frac{|l_1 - l_2|}{3}$

$\lim a_n = \ell_1 - \exists N_1 \quad \ell_1 - \varepsilon \le a_n \le \ell_1 + \varepsilon$ $\quad -\varepsilon \le a_n - \ell_1 \le \varepsilon$

$\forall n \ge N_1$

$\lim a_n = \ell_2 - \exists N_2 \quad \ell_2 - \varepsilon \le a_n \le \ell_2 + \varepsilon$ $\quad -\varepsilon \le a_n - \ell_2 \le \varepsilon$

that is $\quad \forall n \ge N_2$

$-\varepsilon \le a_n - \ell_1 \le \varepsilon \quad \forall n \ge N_1$

$-\varepsilon < a_n - \ell_2 \le \varepsilon \quad \forall n \ge N_2$

that is $\quad |a_n - \ell_1| \le \varepsilon \quad \forall n \ge N_1$

$|a_n - \ell_2| \le \varepsilon \quad \forall n \ge N_2$

both are true $\forall n \ge N = \max\{N_1, N_2\}$

TRIANGULAR INEQUALITY OF MODULUS

$$|x+y| \le |x| + |y|$$

$$\forall x, y \in \mathbb{R}$$

Also on $\mathbb{C}$: if $z_1, z_2 \in \mathbb{C}$

$$|z_1 + z_2| \le |z_1| + |z_2|$$

$$\boxed{|\ell_1 - \ell_2|} = |\underbrace{\ell_1 + a_n}_{x} - \underbrace{a_n - \ell_2}_{y}| <$$

$$|\ell_1 - a_n| + |\ell_2 - a_n| < -$$

for $n \geq N = \max\{N_1, N_2\}$

$$|\ell_1 - a_n| < \varepsilon$$

$$|\ell_2 - a_n| < \varepsilon$$

$$< 2 \cdot \varepsilon = \boxed{\frac{2|\ell_1 - \ell_2|}{3}}$$

contradiction

Theorem: (Permanence of sign)

$\lim_{n\to+\infty} a_n = l > 0$ $(<)$ then

$\exists N$ s.t. $a_n > 0$ $(<)$ $\forall n \geq N$

choose $\varepsilon = \frac{l}{2} > 0$

Proof: $\varepsilon = \frac{l}{2}$

$\exists N$ s.t. $\forall n \geq N$

$$\underbrace{l - \frac{l}{2}}_{\shortparallel} \leq a_n \leq l + \underbrace{\frac{l}{2}}_{\substack{\shortparallel \\ \varepsilon}}$$

$$0 < \frac{l}{2} \leq \boxed{a_n}$$

q.e.d.

Prove the case $l < 0$

Theorem Suppose that a sequence $(a_n)_{n\in\mathbb{N}}$

$\boxed{a_n > 0}$ $(<)$ $\forall n \in \mathbb{N}$

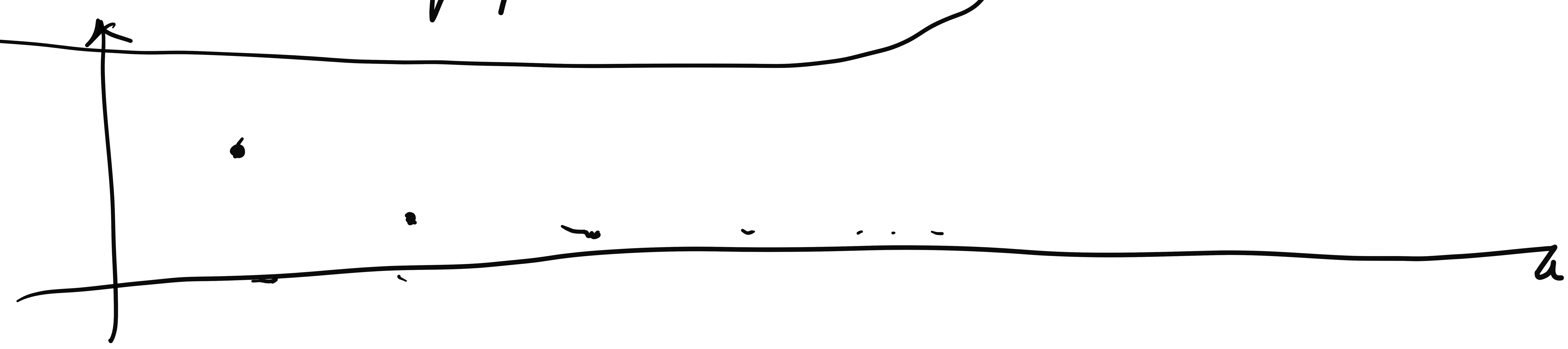
also $\forall n \geq \bar{n}$

and $\boxed{\lim_{n\to+\infty} a_n = l}$

$$\Longrightarrow \boxed{l \geq 0} \quad (\leq)$$

Counter example to $\boxed{l > 0}$

$$a_n = \frac{1}{n} > 0$$



$$\lim_{n\to+\infty} \frac{1}{n} = 0$$

Proof: By contradiction: $l < 0$

$\varepsilon = -\frac{l}{2} > 0 \quad \exists N$ s.t. $\forall n \geq N$

$$\left| \, l - \varepsilon \le \underbrace{a_n \le l + \varepsilon}_{} \, \right|$$

$$a_n \le l - \frac{l}{2} = \frac{l}{2} < 0$$

$$\Longrightarrow a_n < 0$$

contradiction.

q.e.d.