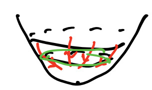
Lesson 9 - 17/10/2022

- Lyapunov theorem for asymptotic stability, with proof.
- Harmonic oscillator with friction: asymptotic stability of $(0,0)$ by an appropriate Lyapunov function and by the first Lyapunov method.
- Rabbits / sheep $\to$ Principle of competitive exclusion.
- $\dot{x} = x^3(x-1)(x+3)$
- $\dot{x} = x^2 + \mu x + 1$, $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$.

LYAPUNOV THEO. for ASYMPTOTIC STABILITY

$\bar{x} \in \mathbb{R}^n$, equilibrium for a v.f. $\dot{x} = X(x)$.
Let suppose that $\exists\ W \in C^1(A; \mathbb{R})$ on $A \ni \bar{x}$ (open neighb. of $\bar{x}$) such that:



① $W(\bar{x}) = 0$ and $W(x) > 0 \quad \forall x \in A \setminus \{\bar{x}\}$.

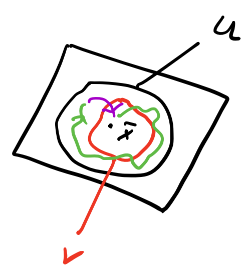
②' $\begin{cases} L_X W(x) < 0 & \forall x \in A \setminus \{\bar{x}\} \\ L_X W(\bar{x}) = 0 \end{cases}$

Then $\bar{x} \in \mathbb{R}^n$ is asympt. stable.

Proof

Condition ②' $\Rightarrow$ ② on topological stability of $\bar{x}$.
So $\bar{x}$ is top. stable. This means that fixed a neigh. $U \ni \bar{x}$ $(U \subseteq A)$ then $\exists\ V \ni \bar{x}$ $(V \subseteq A)$ such that

$$x_0 \in V \Rightarrow \varphi_t(x_0) \in \mathcal{U} \quad \forall t \geq 0.$$



But - in order to prove top asympt. stability -

$$\lim_{t \to +\infty} \varphi_t(x_0) = \bar{x}$$

$$\Leftrightarrow W\left(\lim_{t \to +\infty} \varphi_t(x_0)\right) = W(\bar{x}) \Leftrightarrow$$

$\downarrow$ since W is continuous

$$\Leftrightarrow \lim_{t \to +\infty} W(\varphi_t(x_0)) = W(\bar{x}) = 0$$

$\downarrow$ By assump.

By contrad. suppose that the previous limit doesn't hold.
Since $W(x) > 0 \quad \forall x \in A \setminus \{\bar{x}\}$, this means that

$$\lim_{t\to+\infty} w(\varphi_t(x_0)) = \lambda > 0$$

Now we write this equality by using the def. of limit of a real function for $t \to +\infty$.

EQUALS TO :

$\forall\, \varepsilon > 0 \quad \exists\, T_\varepsilon \geq 0$ such that

if $t \geq T_\varepsilon \Rightarrow |w(\varphi_t(x_0)) - \lambda| \leq \varepsilon$

that is

if $t \geq T_\varepsilon \Rightarrow \lambda - \varepsilon \leq w(\varphi_t(x_0)) \leq \lambda + \varepsilon$

But, recall that $t \mapsto w(\varphi_t(x_0))$ is stricly decreasing!

Then

if $\boxed{t \geq T_\varepsilon} \Rightarrow \lambda \leq w(\varphi_t(x_0)) \leq \lambda + \varepsilon$

$\Downarrow$

λ is the "limit" value. So $t \mapsto w(\varphi_t(x_0))$ cannot assume smaller values that λ.

That is (fixed $\varepsilon > 0$).

$\varphi_t(x_0) \in B := \{ x \in \mathbb{R}^n : \lambda \leq w(x) \leq \lambda + \varepsilon \}$ $\forall\, t \geq T_\varepsilon$

$\lambda > 0$

B is a compact set.

$\bar{x} \notin B$!

As a consequence :

$\max_{x \in B} L_X w(x) \leq -\alpha < 0$

By cond. 2)!

$\Rightarrow L_X w(x) \leq -\alpha < 0 \quad \forall\, x \in B.$

$$L_X w(x) = \frac{d}{dt} w(\varphi_t(x_0))$$

$\Rightarrow w(\varphi^{T_\varepsilon + \tau}(x_0)) - w(\varphi^{T_\varepsilon}(x_0)) \leq -\alpha\tau \quad \forall\, \tau > 0$

$\Rightarrow w(\varphi^{T_\varepsilon + \tau}(x_0)) \leq \underbrace{w(\varphi^{T_\varepsilon}(x_0))}_{\leq \lambda + \varepsilon} - \alpha\tau \quad \forall\, \tau > 0$

$\Rightarrow \omega(\varphi^{T_\varepsilon+\tau}(x_0)) \le \underline{\lambda+\varepsilon-\alpha\tau}, \quad \forall \tau > 0$

But observe now that

$\lambda+\varepsilon-\alpha\tau < \lambda \iff \tau > \varepsilon/\alpha$ $\lightning$ since

$\varphi^t(x_0) \in B \quad \forall\, t \geq T_\varepsilon.$

This is the desired contraddiction. So

$$\lim_{t\to+\infty} \omega(\varphi^t(x_0)) = \underset{>0}{\lambda} \quad \lightning$$

$\Rightarrow$ THE LIMIT MUST BE $=0$. So V is also a basin of attraction $\Rightarrow$ we have the asympt. stability. $\boxtimes$

- $(0,0)$ is an asympt. stable equilibrium for the harmonic oscillator with friction

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\omega^2 x - 2\mu v \end{cases}$$

By using (last week) $E(x,v) = \frac{1}{2}v^2 + \frac{1}{2}\omega^2 x^2$

we obtained

$L_X W(x) = -2\mu v^2 \le 0$ on every neigh of $(0,0)$.

$\downarrow$

X v.f. with friction

$E(x,v)$ proves only simple stability.

$$F(x,v) = \underbrace{\frac{1}{2}v^2 + \frac{1}{2}\omega^2 x^2}_{E(x,v)} + \frac{1}{2}(v+2\mu x)^2 + \frac{1}{2}\omega^2 x^2$$

$F(x,v) > 0$ on a neigh. of $(0,0)$.

$F(0,0) = 0$.

$$L_X F(x,v) = [\omega^2 x + (v+2\mu x)2\mu + \omega^2 x]\,\dot{x} + $$
$$+ [v + (v+2\mu x)]\,\dot{v} = $$

$$= \dots = \underbrace{-2\mu}_{<0}(v^2 + \omega^2 x^2) \leq 0 \quad \text{BUT} = 0 \text{ IFF } (x, v) = (0, 0).$$

$\downarrow$

sub. the exp. for $\begin{cases} \dot{x} = v \\ \dot{v} = -\omega^2 x - 2\mu x \end{cases}$

$\Rightarrow$ $(0,0)$ is ASYMP. STABLE for the harm. oscill. with friction.

OTHER WAY: First Lyap. method !!

$$JX(x,y) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\mu \end{pmatrix}$$

$$\det \begin{pmatrix} -\lambda & 1 \\ -\omega^2 & -2\mu - \lambda \end{pmatrix} = 0$$

$$\Leftrightarrow 2\mu\lambda + \lambda^2 + \omega^2 = 0$$

$$\lambda_{1,2} = \frac{-2\mu \pm \sqrt{4\mu^2 - 4\omega^2}}{2} = -\mu \pm \sqrt{\mu^2 - \omega^2}$$

- $\mu^2 - \omega^2 > 0$ in such a case $\sqrt{\mu^2 - \omega^2} < \mu$
  $\Rightarrow$ 2 real < 0 eigenvalues!
- $\mu^2 = \omega^2 \Rightarrow \lambda_{1,2} = -\mu < 0$
- $\mu^2 - \omega^2 < 0 \Rightarrow$ 2 complex conjug. eigenvalues with < 0 real part.

THE COND. FOR THE FIRST LYAPUNOV METHOD ALWAYS HOLD!!!

$\Downarrow$

ASYMP. STABILITY!

— x —

When rabbits and sheep encounter each other, trouble starts. Since the two populations are in competition for food, we assume that the conflicts occur at a rate proportional to the size of each population.

Furthermore, we assume that the conflicts reduce the growth rate for each species, but the effect is more severe for rabbits.

$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - y - x) \end{cases}$$

$x(t) =$ pop. of rabbits
$y(t) =$ pop. of sheep.
$x, y \geq 0$

Study the dynamics.

SOLUTION

First step. EQUILIBRIA

$$\begin{cases} x(3 - x - 2y) = 0 \\ y(2 - y - x) = 0 \end{cases}$$

First case

$$\begin{cases} x = 0 \\ y(2 - y) = 0 \end{cases} \to (0,0) \text{ OR } (0, 2)$$

other case

$$\begin{cases} 3 - x - 2y = 0 \to x = 3 - 2y \\ y(2 - y - 3 + 2y) = 0 \end{cases} \iff \begin{cases} x = 3 - 2y \\ y(y - 1) = 0 \end{cases}$$

$\iff (3, 0)$ OR $(1, 1)$

Four equilibria for the non-linear v.f.

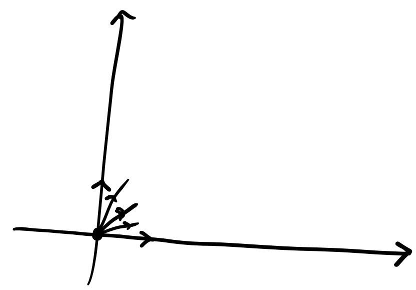
$(0,0), (0, 2), (3,0), (1, 1)$

$$A = \begin{pmatrix} 3 - x - 2y - x & -2x \\ -y & 2 - x - y - y \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}$$

$$A(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \to (0,0) \text{ is an unstable node}$$

$\lambda_1 = 2 \to v_1 = (0, 1)$

$\lambda_2 = 3 \to v_2 = (1, 0)$



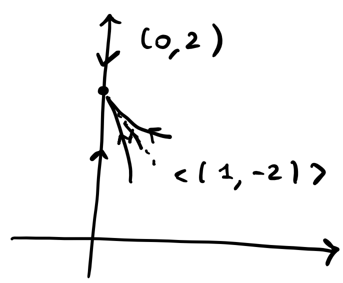
$$A(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$$

with eigenvalues

$\lambda_1 = -1$, $v_1 = (1, -2)$

$\lambda_2 = -2$, $v_2 = (0, 1)$

$(0,2)$ is a stable node!

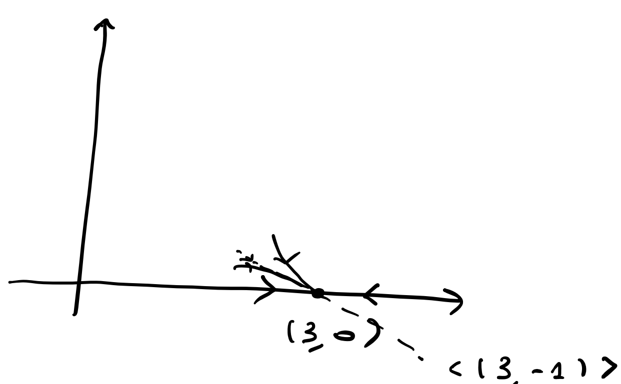


$$A(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$

$\lambda_1 = -3$ and $v_1 = (1, 0)$

$\lambda_2 = -1$ and $v_2 = (3, -1)$

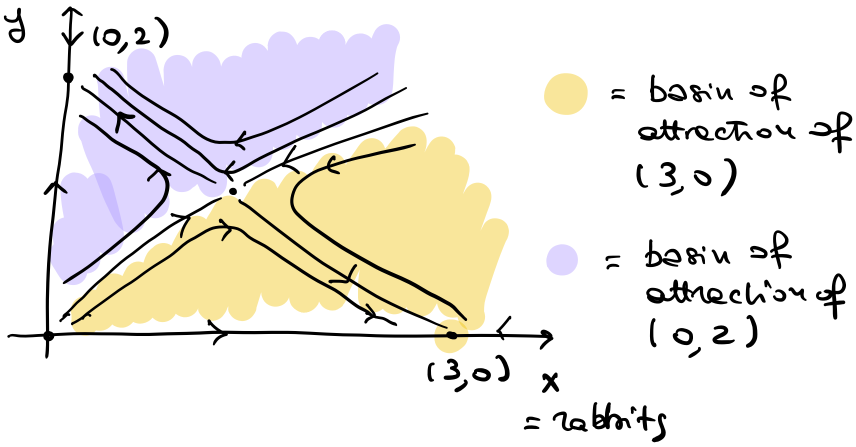
$(3,0)$ is a stable node.



$$A(1,1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

$\det = 1 - 2 < 0 \Rightarrow$ saddle point! $\quad (\lambda_{1,2} = -1 \pm \sqrt{2})$

sheep $= y$



Essentially in every case, one species drives the other to extinction ! $\to$ BIOLOGICAL INTERPRETATION.

$\Downarrow$

PRINCIPLE of COMPETITIVE EXCLUSION :

2 species competiting for the SAME limited resourse cannot coexist !!

$\boxed{\text{EX } 1}$ $\quad \dot{x} = x^3 (x-1)(x+3)$

1) Find equilibria

2) Linearize the v.f. around the ep. $x_0 > 0$ and discuss its stability.

3) Can we use x^2 as lyap. function to prove the stability of $x_0 = 0$?

SOL $\quad x^3 (x-1)(x+3) = 0 \;\to\; x = 0, \; x = 1, \; x = -3$

$$X'(x) = 3x^2 (x-1)(x+3) + x^3 (x+3) + x^3 (x-1)$$

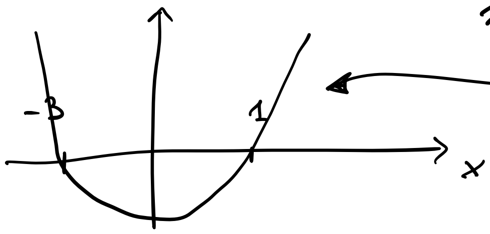
$$\Rightarrow X'(1) = \underset{>0}{4} \Rightarrow \underline{\dot{x} = 4(x-1)}$$

$\Rightarrow$ 1 is unstable.

$X'(0) = 0 \Rightarrow$ No infos on 0 by the first derivative.

$W(x) = x^2$ is a good candidate Lyapunov function for $x_0 = 0$.

$$L_X W(x) = 2x\dot{x} = \underbrace{2x^4}_{\geq 0}\underbrace{(x-1)(x+3)}_{\text{parabola.}} < 0 \quad \text{in }]-3, 1[\setminus\{0\}.$$



$\Rightarrow$ 0 is ASYMP. STABLE!

$\boxed{\text{EX2}}$ Bifurcation diagram for $\dot{x} = x^2 + \mu x + 1$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$.

$\boxed{\text{SOL}}$ $X_\mu(x) = x^2 + \mu x + 1 = 0$.

$$x_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 - 4}}{2} \iff \mu^2 - 4 \geq 0$$

$\iff \mu \leq -2$ or $\mu \geq 2$.

In particular,

if $\mu \in (-\infty, -2) \cup (2, +\infty)$: 2 real, distinct equilibria.

if $\mu = \pm 2$, we have $x_1 = x_2$

if $\mu \in (-2, 2)$, NO equilibria.

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