$P(z) = a_n z^n + \dots + a_1 z + a_0$

Complex polynomial

$a_0, \dots, a_n \in \mathbb{C}$

Lemma (Ruffini's) $w \in \mathbb{C}$ is a <u>solution</u> of $P(z)=0$ if and only if

$$P(z) = (z-w)\, Q(z)$$

with $Q(z)$ polynomial $n-1$.

Theorem (Fund. Th. of algebra) If $P(z)$ is a polynomial of degree n then there exist $\mathbb{C} \ni z_1, \dots, z_N$ with $N \le n$ and $c_1, \dots, c_N \in \mathbb{N} \setminus \{0\}$ such that

$$P(z) = a_n (z-z_1)^{c_1} (z-z_2)^{c_2} \cdots (z-z_N)^{c_N}$$

$c_1 + \dots + c_N = n$

We call c_k the <u>multiplicity of z_k</u>

Suppose

$$P(z) = a_n z^n + a_{n-1} z^{n-1} \cdots + a_1 z + a_0$$

$a_0, a_1, \ldots, a_n \in \mathbb{R}$

Proposition: If $P(z)$ has real coefficients, then

w is a soln. of $P(z) = 0$

$\Updownarrow$

$\overline{w}$ is a soln of $P(z) = 0$

"$\Downarrow$" w is a solt. $\Longleftrightarrow P(w) = 0$

$\Longleftrightarrow \underbrace{a_n w^n + \cdots\cdots + a_1 w + a_0}$ $= 0$

$\Longleftrightarrow \overline{a_n w^n + \cdots + a_1 w + a_0} = 0$

$\Longleftrightarrow \overline{a_n w^n} + \cdots + \overline{a_1 w} + \overline{a_0} = 0$

$a_n (\overline{w})^n + a_{n-1} (\overline{w})^{n-1} \cdots + a_1 \overline{w} + a_0 = 0$

$\shortparallel$

$P(\overline{w}) = 0$

$$\boxed{\begin{aligned} \overline{w_1 + w_2} &= \overline{w_1} + \overline{w_2} \\ \overline{w_1 \cdot w_2} &= \overline{w_1}\,\overline{w_2} \\ \overline{(w^k)} &= (\overline{w})^k \end{aligned}}$$

$$P(z) = a_n z^n + \cdots + a_0$$

by F.T.A we know that

$$P(z) = a_n (z - z_1)^{c_1} \cdots (z - z_N)^{c_N}$$

$$c_1 + \cdots + c_N = n$$

Suppose that z_k is a non-real solution for some $k = 1, \ldots, N$

we know that $\overline{z}_k$ must be a solution

$$Q_k(z) = (z - z_k)(z - \overline{z}_k) =$$

$$z^2 + (-z_k - \overline{z}_k) z + z_k \overline{z}_k =$$

$$= \underbrace{z^2 - 2\,\mathrm{Re}(z_k) \cdot z + |z_k|^2}_{\text{This is a 2-degree real polynomial}}$$

$$P(z) = a_n (z - z_1)^{c_1} \cdots (z - z_N)^{c_N} =$$

$$= a_n (z - z_1)^{c_1} \cdots (z - z_{N_1})^{c_{N_1}} Q_{N_1+1}^{c_{N_1+1}} \cdots Q_N^{c_N}$$

$\underbrace{z_1 \ldots z_{N_1}}_{\in \mathbb{R}}, \underbrace{z_{N_1+1} \ldots z_N}_{\in \mathbb{C} \setminus \mathbb{R}}$

with Q_k 2-degrees REAL polinomials

We have proved that

<u>Theorem</u> If $P(z)$ has real coefficients then it can be factorize into <u>real</u> polinomials of degre 1 and 2.

$x \in \mathbb{R}$ $P(x) = x^3 + x = \boxed{\underbrace{x}\underbrace{(x^2+1)}}$

the best factorization

$z \in \mathbb{C}$

$$P(z) = z^3 + z = z(z^2+1) =$$

$$z\underbrace{(z+i)(z-i)}_{\substack{z^2 - i^2 \\ \parallel \\ z^2 + 1}}$$

<u>Exercise</u>. Find the roots

of $P(z) = z^4 + z^3 - 11z^2 + z - 12 =$

1) verify that 3 and -4 are roots

2) Using 1), find all roots

Sol.

i) $P(3) = 81 - 27 - 11 \cdot 9 + 3 - 12 =$

$= 84 + 27 - 99 - 12 = 0$

$P(-4) = 0$

$P(z) = \underbrace{(z-3)(z+4)}\, Q(z)$

$$
\begin{array}{l|l}
z^4 + z^3 - 11z^2 + z - 12 & z^2 + z - 12 \\
\hline
z^4 + z^3 - 12z^2 & z^2 + 1 \\
\hline
z^2 + z - 12 & \\
z^2 + z - 12 & \\
\hline
0 &
\end{array}
$$

$P(z) = (z-3)(z+4)(z^2+1) =$

$(z-3)(z+4)(z+i)(z-i)$

$z_1 = 3$

$z_2 = -4$

$z_3 = -i$

$z_4 = i$

$$z^8 + iz^7 + i2z^5 - \lambda z^4 = P(z)$$

1) Find $\lambda \in \mathbb{R}$ such that $-i$ is a root of $P(z)$

2) For this λ find all roots of $P(z)$

$$(-i)^7 = (-i)^4(-i)^3 = = (-i)^3 = -1(-i) = i$$

1) $P(-i) = 0$

$$1 + i(i) + i2(-i) - \lambda = 0$$

$$1 - 1 + 2 - \lambda = 0 \iff \lambda = 2$$

$$P(z) = z^8 + iz^7 + 0z^6 + i2z^5 - 2z^4 + 0z^3 + 0z^2 + 0z + 0 \quad \Big| \; z+i$$

$$z^8 + iz^7$$

$$i2z^5 - 2z^4 + \dots$$

$$i2z^5 - 2z^4$$

$$0$$

$$z^7 + i2z^4$$

$$P(z) = (z+i)(z^7 + i2z^4)$$

$$P(z) = 0 \iff z + i = 0 \text{ or } z^7 + i2z^4 = 0$$

$z = x + iy$ $\quad (x+iy)^7 + i2(x+iy)^4 = 0$

$$\underline{(x+iy)^4}\left((x+iy)^3 + i2\right) = 0$$

$x + iy = 0 \Longleftrightarrow x = 0 - y = 0 \quad z_1 = 0$

root with multiplicity

$c_1 = 4$

or

$(x+iy)^3 + i2 = 0$

$z^3 + i2 = 0$

$z^3 = -i2$

$-i2 = 2e^{i\frac{3}{2}\pi}$

$z_0 = \sqrt[3]{2}\, e^{i\frac{\pi}{2}} = \sqrt[3]{2}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)$

$\sqrt[3]{2}\,(i)$

$z_4 = \sqrt[3]{2}\, e^{i\left(\frac{\pi}{2} + \frac{2}{3}\pi\right)} =$

$z_8 = \sqrt[3]{2}\, e^{i\left(\frac{\pi}{2} + \frac{4}{3}\pi\right)} =$

Plot in the Gauss plane $\left\{ \operatorname{Im}\left(\frac{i(z-3)}{z-1}\right) < 0 \right\}$

$z = x + iy$

$$\operatorname{Im}\left(i \frac{((x-3)+iy)\,[(x-1)-iy]}{[(x-1)+iy]\,[(x-1)-iy]} \right) =$$

$$\operatorname{Im}\left(\frac{-y(x-1)+(x-3)y+i\left((x-3)(x-1)+y^2\right)}{(x-1)^2+y^2} \right) < 0$$

$$\Updownarrow$$

$$\frac{(x-3)(x-1)+y^2}{(x-1)^2+y^2} < 0$$

$x \neq 1$
$y \neq 0$
$z \neq 1$

$$\Updownarrow$$

$$(x-3)(x-1)+y^2 < 0$$

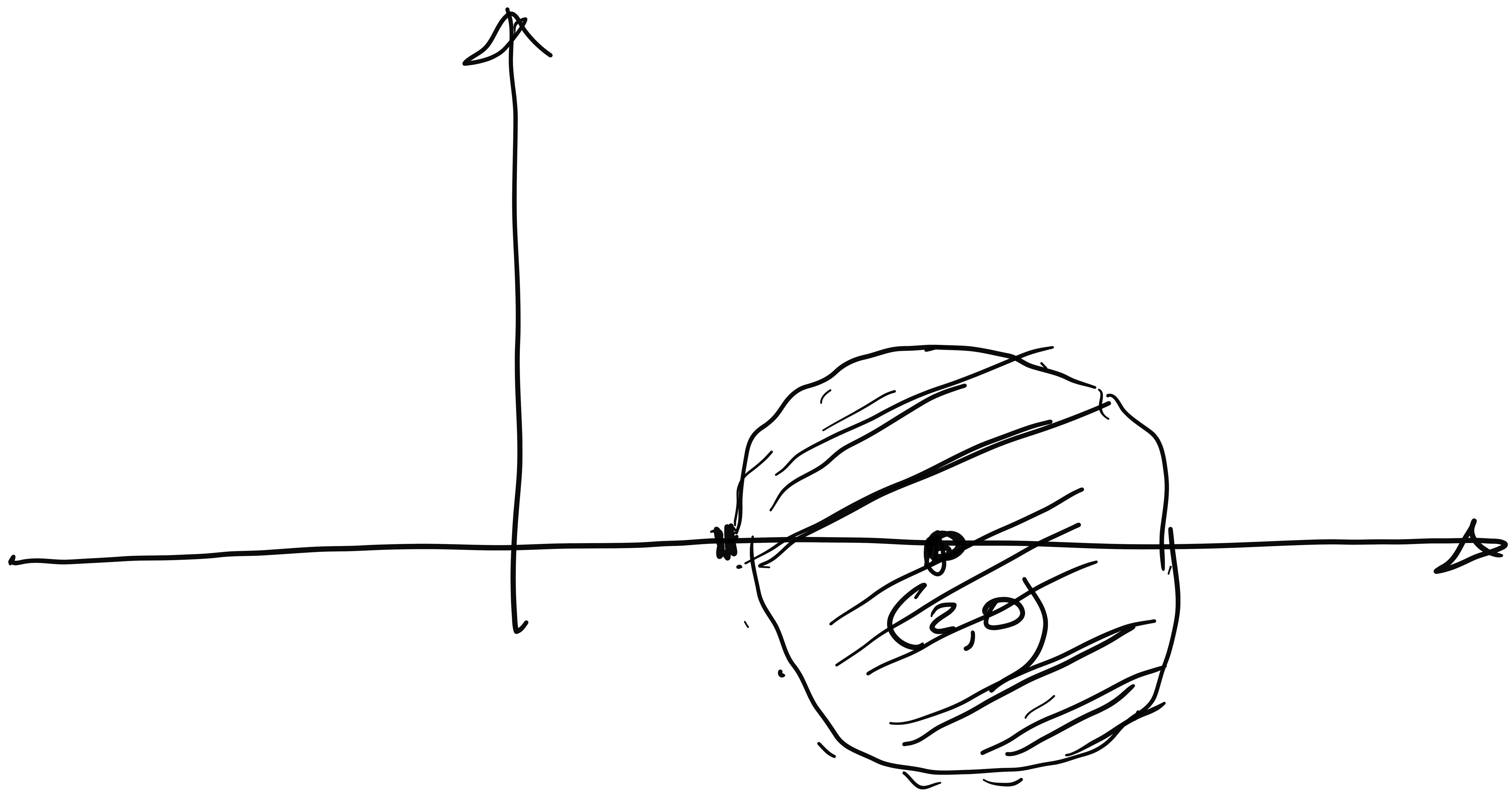
$$x^2-4x+3+y^2 < 0$$

$$\underbrace{x^2+y^2-4x+3} < 0$$

$x^2+y^2-4x+3 = 0$ circle

center of circle $(2,0)$

$r = 1$


(2,0)