# Lesson 8 - 13/10/2022

## From linear to general non-linear vector fields

(on $\mathbb{R}^n$) $\dot{z} = X(z)$ with $X(\bar{z}) = 0$ $\quad \rightsquigarrow \quad$ $\dot{u} = Au$ $\quad$ $u = z - \bar{z}$, $\quad A = \frac{\partial X}{\partial z}(\bar{z})$

$V_{\bar{z}}$ neigh. of $\bar{z}$ $\qquad\qquad$ $V_0$ neigh. of $0$

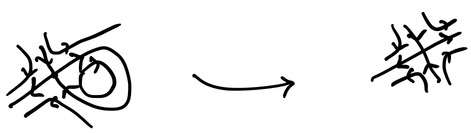
**Question / hope**

Does exist a local diffeo. $\kappa : V_{\bar{z}} \to V_0$ (= local change of variables) mapping the solutions of $\dot{z} = X(z)$ into the solutions of $\dot{u} = Au$. That is:

$$\kappa(\phi^t(z)) = e^{tA}\kappa(z) \qquad \forall z \in V_{\bar{z}},\ \forall t \in \mathbb{R} \text{ such that } \varphi^t(z) \in V_{\bar{z}}$$

**Quasi...**

- (Grobman - Hartman Theorem) (only statement)
  Let $\bar{z} \in \mathbb{R}^n$ be an hyperbolic eq. of $\dot{z} = X(z)$. Then $\dot{z} = X(z)$ and $\dot{u} = Au$ are LOCALLY TOPOLOGICALLY EQUIVALENT that is $\kappa$ is an homeomorphism.



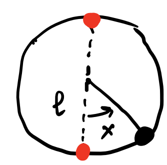
- Other related results
  FIRST LYAPUNOV THEOREM.
  $\bar{z} \in \mathbb{R}^n$ s.t. $X(\bar{z}) = 0$
  i) If $A = \frac{\partial X}{\partial z}(\bar{z})$ has ALL eigenvalues with strictly negative real part $(<0)$ then $\bar{z}$ is "ASYMPTOTICALLY STABLE".
  ii) If $A = \frac{\partial X}{\partial z}(\bar{z})$ has AT LEAST ONE eigenvalue with strictly positive real part $(>0)$, then $\bar{z}$ is "UNSTABLE".

## EQUILIBRIA & STABILITY

$\dot{x} = X(x)$, $\quad X \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n)$

① $\bar{x}$ is an eq. if $X(\bar{x}) = 0 \iff$ there exists the constant solution $\varphi^t(\bar{x}) \equiv \bar{x} \quad \forall t \in \mathbb{R}$.

② Pendulum $\quad \ddot{x} + \omega^2 \sin x = 0 \quad (\omega^2 = g/\ell)$




$$\begin{cases} \dot{x} = v \\ \dot{v} = -\omega^2 \underbrace{\sin x}_{= 0} \end{cases}$$

EQUILIBRIA ? $(0,0)$
$(\pi, 0)$

Pendulum with friction $\quad \ddot{x} = -\omega^2 \sin x - \underbrace{\mu \dot{x}}_{\mu > 0}$

$$\begin{cases} \dot{x} = v = 0 \\ \dot{v} = -\omega^2 \sin x - \underbrace{\mu v}_{= 0} \end{cases}$$

EQUILIBRIA $(0,0)$, $(\pi, 0)$ are the same

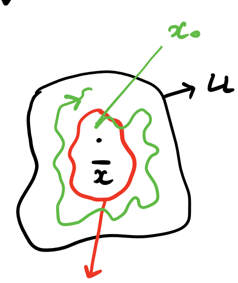
BUT $(0,0)$ is QUALITATIVE different!

$\boxed{\text{DEF}}$ (STABILITY)

An eq. $\bar{x} \in \mathbb{R}^n$ for the v.f. $X(x)$ is called (TOPOLOGICALLY) STABLE (in the future) if for every neigh. $\mathcal{U}$ of $\bar{x}$ in $\mathbb{R}^n$ there exists a neigh $V$ of $\bar{x} \in \mathbb{R}^n$ s.t.

$$\varphi^t(x_0) \in \mathcal{U} \qquad \forall x_0 \in V, \ \forall t \geq 0$$

$[x_0 \in V \Rightarrow \varphi^t(x_0) \in \mathcal{U} \quad \forall t \geq 0]$

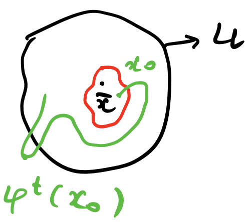



(UNSTABILITY)

An eq. $\bar{x} \in \mathbb{R}^n$ for the v.f. $X(x)$ is called UNSTABLE if it is not topologically stable.

That is:

$\exists\, \mathcal{U} \ni \bar{x}$ s.t. $\forall\, V$ neigh. of $\bar{x}$, $\exists\, x_0 \in V$ and $\exists\, t > 0$ such that $\varphi^t(x_0) \notin \mathcal{U}$.



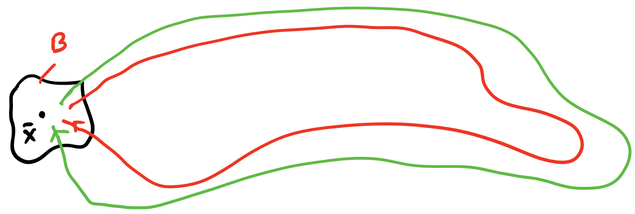

(ASYMPTOTICALLY STABILITY)

An eq. $\bar{x} \in \mathbb{R}^n$ for the v.f. $X(x)$ is called asymptotically stable if

1) $\bar{x}$ is (topologically) stable.

2) $\exists\, B \ni \bar{x}$ neigh. of $\bar{x}$ s.t. $\lim_{t \to +\infty} \varphi^t(x_0) = \bar{x} \quad \forall x_0 \in B$.

$B$ = basin of attraction.




THEOREM (Lyapunov, 1892) (topological version)

Let $\bar{x} \in \mathbb{R}^n$, eq. of $\dot{x} = X(x)$.

Let suppose that there exists $W \in C^0(A; \mathbb{R})$ (A open neigh. of $\bar{x}$)

$W : A \to \mathbb{R}$, continuous.

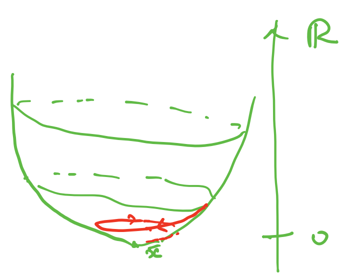
$\bar{x}^t$

such that

① $W(\bar{x}) = 0$ and $W(x) > 0 \quad \forall x \in A \setminus \{\bar{x}\}$.

② $t \mapsto W(\varphi^t(x_0))$ is non-increasing on $A$.

Then $\bar{x}$ is (topologically) stable.



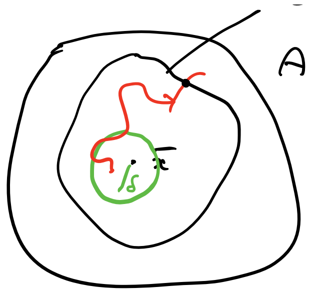

PROOF

Fix a neigh. $U \ni \bar{x}$

$U \subseteq A$

U

$W \in C^0(A;\mathbb{R}) \Rightarrow W \in C^0(\underbrace{\partial \mathcal{U}}_{\text{compact}};\mathbb{R})$




$\Rightarrow$ There exists (by Weierstrass theorem)

$$a = \min_{x \in \partial \mathcal{U}} W(x) \underbrace{> 0}_{\text{by cond. 1}}$$

Now we write in detail continuity of $W$ in $\bar{x}$, by using $a$ "as $\varepsilon > 0$" ← the

That is, corresponding to $a > 0$ $\exists\, \delta > 0$ such that

$$x_0 \in B(\bar{x}, \delta) \Rightarrow \underbrace{|\overbrace{W(x_0)}^{>0} - \underset{\underset{0}{\parallel}}{W(\bar{x})}| < a}$$

$$( W(x_0) < a )$$

We prove now that $B(\bar{x}, \delta) = V$. Let $x_0 \in B(\bar{x}, \delta)$

Take $\varphi^t(x_0)$ and use hypothesis 2).

If $\varphi^t(x_0) \notin \mathcal{U}$ for some time $t > 0 \Rightarrow \exists\, \bar{t} > 0$ such that $\varphi^{\bar{t}}(x_0) \in \partial \mathcal{U}$ that is $W(\varphi^{\bar{t}}(x_0)) \geq \underset{(\min W \text{ on } \partial\mathcal{U})}{\overset{}{\underset{\parallel}{a}}}$

But this is impossible since, by hypothesis 2), $t \mapsto W(\varphi^t(x_0))$ is non increasing on $A$.

$\boxtimes$

## Remark

Cond. 2) requires the knowledge of solutions ... $\varphi^t(x_0)$ ...

But, if we know solutions explicitly, we can make conclusions about stability without a Lyapunov function!!

This "pessimism" can be removed by considering

$$\underline{W \in C^1(A;\mathbb{R})}.$$

Let $x_0 \in A$

$t \mapsto W(\varphi^t(x_0))$ is non-increasing in $A$ iff

$\partial_t(W(\varphi^t(x_0))) \leq 0$ on $A$ iff $\nabla W(\varphi^t(x_0)) \cdot X(\varphi^t(x_0)) \leq 0$

on $A$ then $(t=0)$ $\boxed{\nabla W(x_0) \cdot X(x_0) \leq 0 \quad \forall\, x_0 \in A.}$

On the other hand, if $\nabla w(x_0)\cdot X(x_0) \leq 0 \quad \forall x_0 \in A$

then $\nabla w(\varphi^t(x_0))\cdot X(\varphi^t(x_0)) \leq 0$ on $A$

that is $t \mapsto w(\varphi^t(x_0))$ is non increasing on $A$.

$\boxed{2} \iff \nabla w(x)\cdot X(x) \leq 0 \quad \forall x \in A$

$\downarrow \qquad \boxed{w \in C^1}$

Theo.

**DEF** Lie Derivative (of a $C^1$ function on a v.f.)

$w \in C^1(\mathbb{R}^n; \mathbb{R})$, $X \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$

$$(L_X w)(x) := \nabla w(x)\cdot X(x)$$

**THEOREM** Lyapunov 1892 - differential version -

$\bar{x} \in \mathbb{R}^n$ eq. of $\dot{x} = X(x)$.

Suppose $\exists\ w \in C^1(A; \mathbb{R})$ $\quad \bar{x} \in A$

such that

① $w(\bar{x}) = 0$ and $w(x) > 0 \quad \forall x \in A \setminus \{\bar{x}\}$

② $(L_X w)(x) \leq 0 \quad \forall x \in A$

then $\bar{x}$ is (topologically) stable.

$\boxed{\text{EX}}$

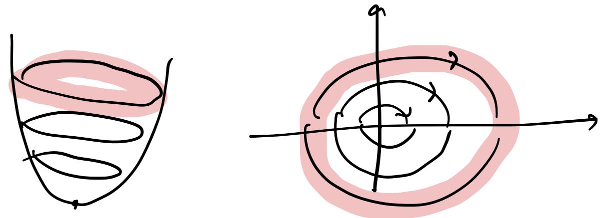
Lyapunov function for the harmonic oscillator with friction

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\omega^2 x - 2\mu v \end{cases} \qquad (\mu > 0)$$

$(0,0) \to$ Asymp. stable!

In the case $\mu = 0$ (standard harmonic oscillator),

$E(x,v) = \frac{1}{2}v^2 + \frac{1}{2}\omega^2 x^2$ (conserved quantity, first integral, $L_X E \equiv 0$)



$$\begin{cases} \dot{x} = v \\ \dot{v} = -\omega^2 x \end{cases}$$

$$L_X E(x,v) = \nabla E(x,v) \cdot X(x,v) =$$

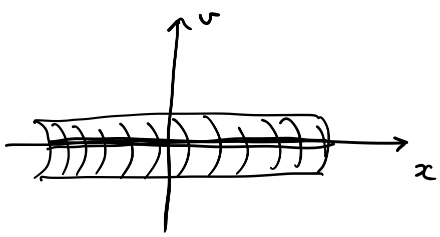
$$= (\omega^2 x, v) \cdot \begin{pmatrix} v \\ -\omega^2 x \end{pmatrix} \equiv 0$$

Use $E(x,v)$ as a Lyap. function in the case with friction.

$$L_{X} E(x,v) = (\omega^2 x, v) \cdot \begin{pmatrix} v \\ -\omega^2 x - 2\mu v \end{pmatrix} =$$

$X$ → v.f. with friction

$$= \cancel{\omega^2 x v} - \cancel{\omega^2 x v} - 2\mu v^2 = -2\mu v^2 \leq 0 \to \text{Stability.}$$




$$F(x,v) = \underbrace{\frac{1}{2} v^2 + \frac{1}{2}\omega^2 x^2}_{E(x,v)} + \frac{1}{2}(v^2 + 2\mu x)^2 + \frac{1}{2}\omega^2 x^2$$

—x—x—