$\hat{z} \in \mathbb{C}$ $\quad z = \rho e^{i\theta} = \boxed{\rho(\cos\theta + i\sin\theta)}$

$\hat{z} \neq 0$

Theorem (De Moivre) $\forall n \in \mathbb{N}$

$\exists$ exactly n nth-roots of $\hat{z}$

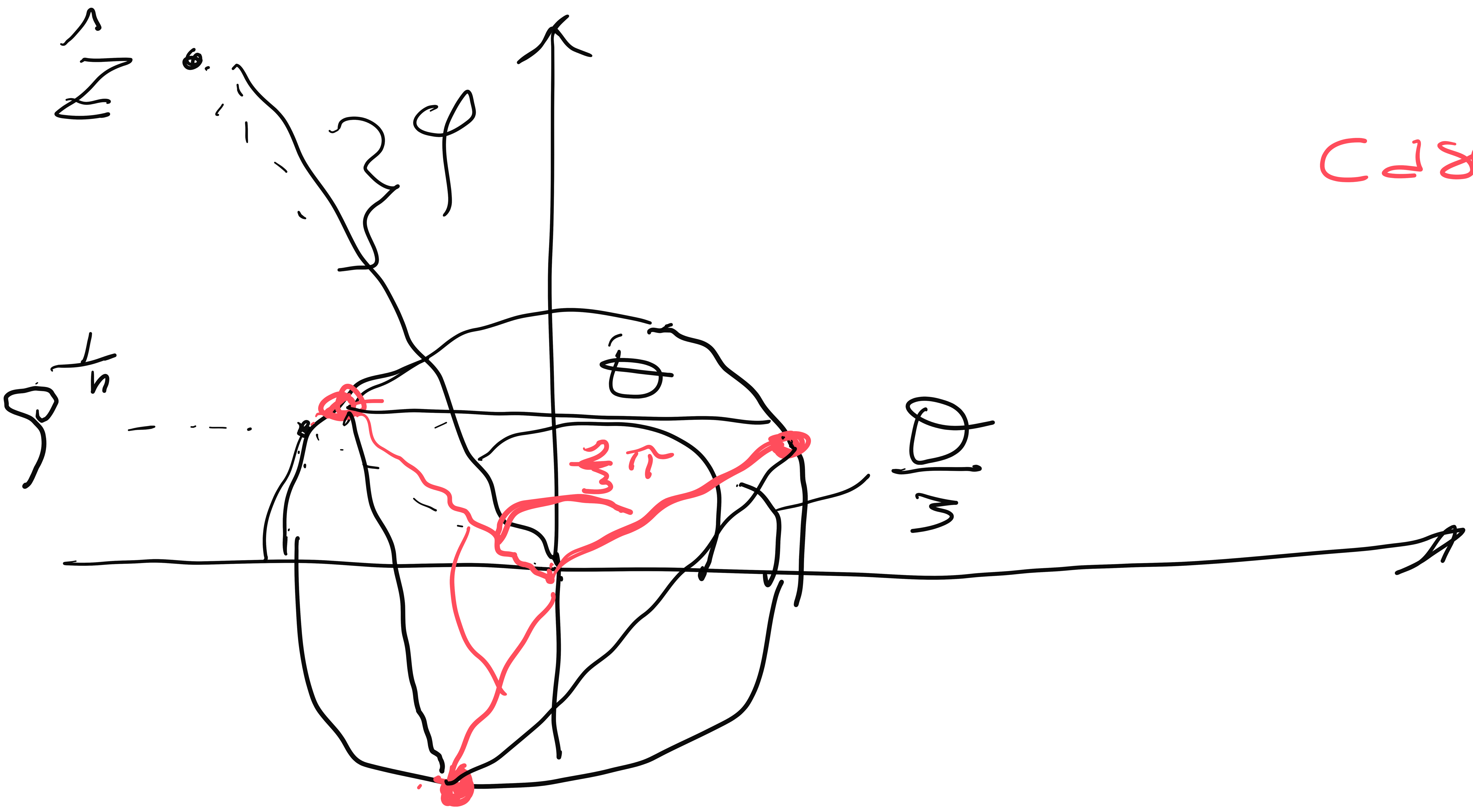
$$z_0 = \rho^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right)$$

$$z_1 = \rho^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n} + \frac{2\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi}{n}\right)\right)$$

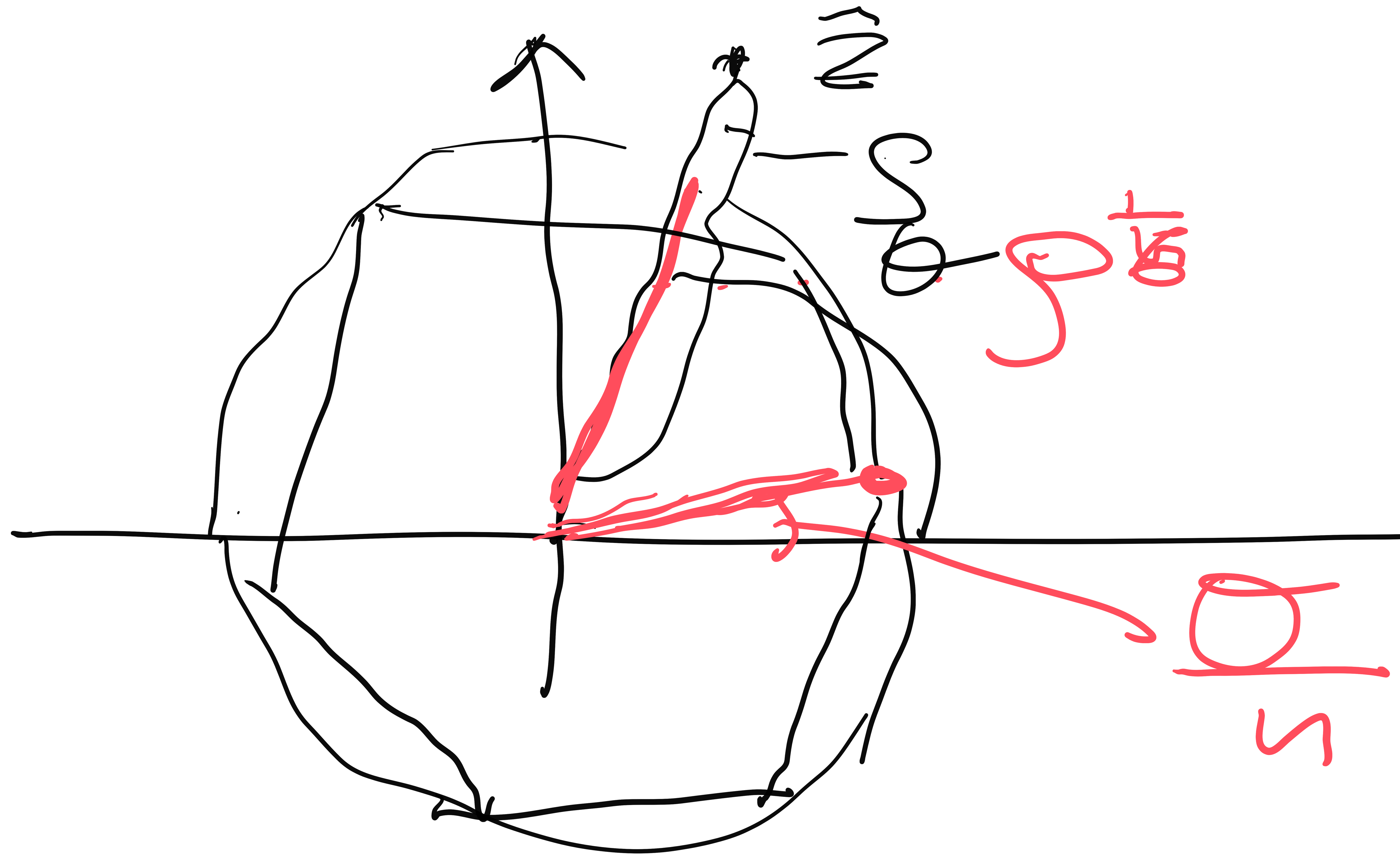
$k = 0, \dots, n-1$

$$z_k = \rho^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n} + \frac{k\,2\pi}{n}\right) + \sin\left(\frac{\theta}{n} + \frac{k\,2\pi}{n}\right)\right)$$

$$z_{n-1} = \rho^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n} + \frac{(n-1)2\pi}{n}\right) + \sin\left(\frac{\theta}{n} + \frac{(n-1)2\pi}{n}\right)\right)$$



case $n = 3$



$n=6$

Proof: $\boxed{z^n = \hat{z}}$ (*)

$$z = r(\cos\varphi + i\sin\varphi)$$

$$z^n = r^n\left(\cos(n\varphi) + i\sin(n\varphi)\right) =$$

$$\hat{z} = \rho(\cos\theta + i\sin\theta)$$

$$\Updownarrow$$

$$r^n = \rho \iff r = \rho^{\frac{1}{n}} = \sqrt[n]{\rho}$$

$$n\varphi_0 = \theta \iff \varphi_0 = \frac{\theta}{n}$$

$$n\varphi_1 = \theta + 2\pi \iff \varphi_1 = \frac{\theta}{n} + \frac{2\pi}{n}$$

$$n\varphi_k = \theta + 2k\pi \iff \varphi_k = \frac{\theta}{n} + \frac{2k\pi}{n}$$

$$z_n = \rho^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n} + \frac{\cancel{n}2\pi}{\cancel{n}}\right) + i\sin\left(\frac{\theta}{n} + \frac{\cancel{n}2\pi}{\cancel{n}}\right)\right) = z_0$$

$$z_{n+1} = \ldots = z_1$$

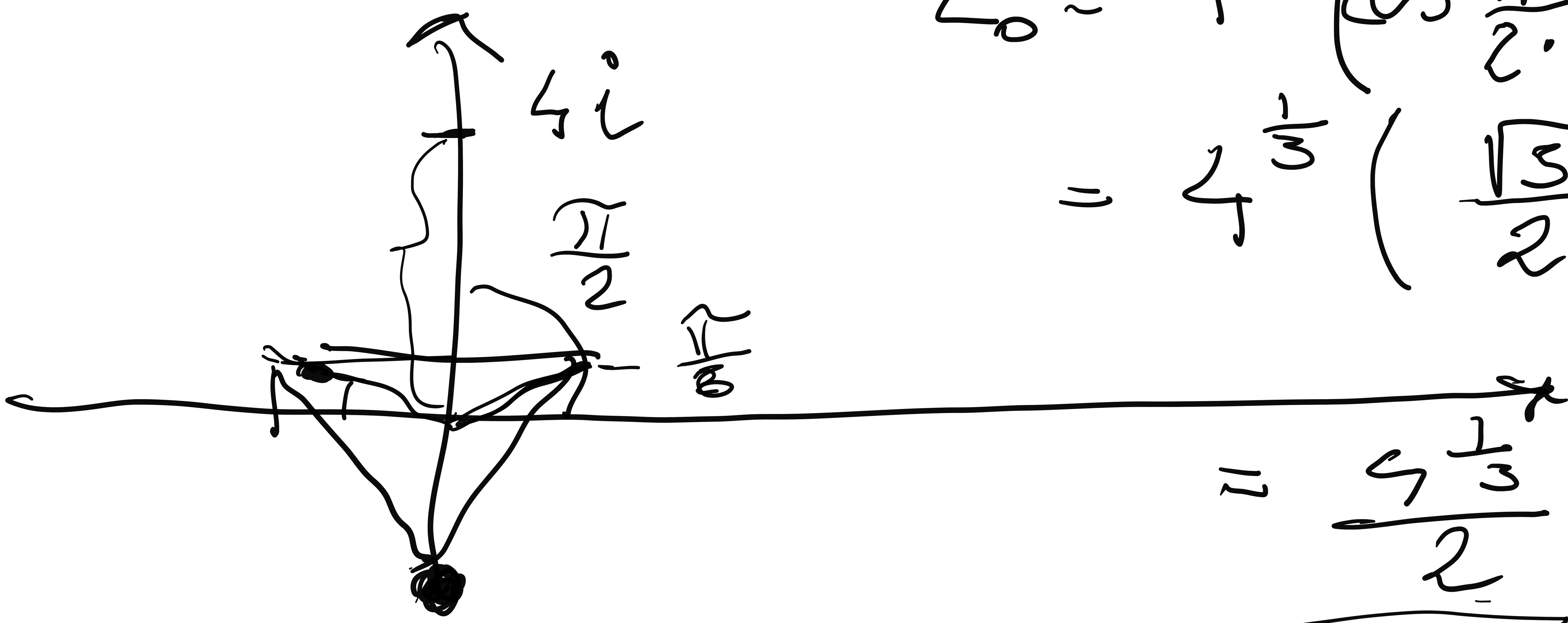
q.e.d.

Exercise $\hat{z} = 4i$

Find the third roots of $4i$

$$4i = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$z_0 = 4^{\frac{1}{3}}\left(\cos\frac{\pi}{2\cdot3} + i\sin\frac{\pi}{2\cdot3}\right) = 4^{\frac{1}{3}}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \frac{4^{\frac{1}{3}}}{2}\left(\sqrt{3} + i\right)$$



$$\frac{\pi}{6} + \frac{2}{3}\pi = \frac{5}{6}\pi$$

$$\boxed{z_1 = 4^{\frac{1}{3}}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$$

$$= 4^{\frac{1}{3}}\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$= \frac{4^{\frac{1}{3}}}{2}\left(-\sqrt{3} + i\right)$$

$$\boxed{z_2 = 4^{\frac{1}{3}}(-i) = -i\,4^{\frac{1}{3}}}$$

Find the sixth roots of $4i$.

$$z_0 = 4^{\frac{1}{6}}\left(\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right)$$

- $$\cos(\psi) = \sqrt{\frac{1+\cos(2\psi)}{2}}$$
- $$\sin(\psi) = \sqrt{\frac{1-\cos(2\psi)}{2}}$$

$$\cos\left(\frac{\pi}{12}\right) = \sqrt{\frac{1+\cos\frac{\pi}{6}}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

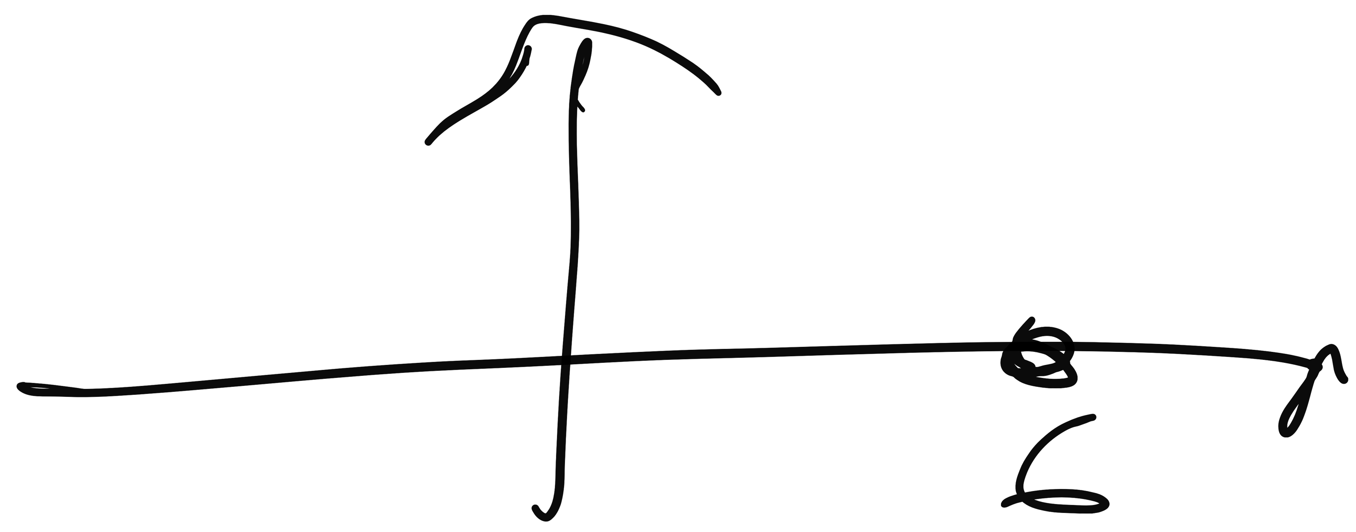
$$\sin\left(\frac{\pi}{12}\right) = \sqrt{\frac{1-\cos\frac{\pi}{6}}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$z_0 = 4^{\frac{1}{6}}\left(\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} + i\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\right)$$

$z = 6$ find square-roots

$z_0 = \sqrt{6}$

$z_1 = -\sqrt{6}$



$$6 = 6(\cos 0 + i \sin 0)$$

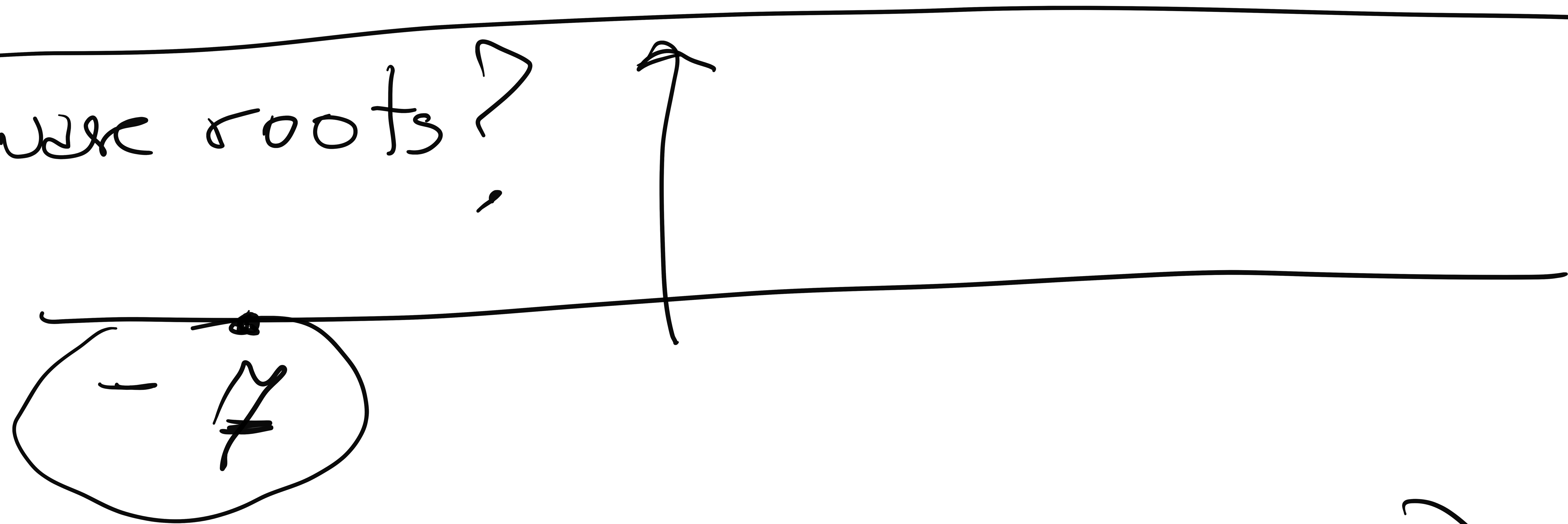
$$z_0 = \sqrt{6}\left(\cos \frac{0}{2} + i \sin \frac{0}{2}\right) =$$

$$= \sqrt{6}\,(1) = \sqrt{6}$$

$$z_1 = \sqrt{6}\left(\cos\left(\frac{0}{2} + \frac{2\pi}{2}\right) + i \sin\left(\frac{0}{2} + \frac{2\pi}{2}\right)\right)$$

$$= \sqrt{6}\,(-1 + i0) = -\sqrt{6}$$

Square roots?



$$-7 = 7(\cos \pi + i \sin \pi)$$

$$z_0 = \sqrt[2]{7}\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) =$$

$$= \sqrt[2]{7}\,(i) = i\sqrt[2]{7}$$

$$z_1 = \sqrt[2]{7}\left(\cos\left(\frac{\pi}{2}+\pi\right) + i\sin\left(\frac{\pi}{2}+\pi\right)\right) =$$

$$= \sqrt[2]{7}\,(0 - i) = -i\sqrt[2]{7}$$

$\alpha \in \mathbb{C}$ $$\underbrace{z^n + \alpha}_{P(z)} = 0$$

$$P(z) = 0 \quad (*)$$

The solutions are also called the roots of P.

$$P(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \dots + \alpha_1 z + \alpha_0$$

$\alpha_0 \dots \alpha_n \in \mathbb{C}$

The solutions of

$$P(z) = 0$$

are called the roots of $P(z)$

Particular case

$$P(z) = a_2 z^2 + a_1 z + a_0$$

$a_0, a_1, a_2 \in \mathbb{R}$

$$a_2 z^2 + a_1 z + a_0 = 0$$

Real solutions:

$$\boxed{z_0 = \frac{-a_1 + \sqrt{\Delta}}{2a_2} \qquad z_1 = \frac{-a_1 - \sqrt{\Delta}}{2a_2}}$$

provided $\Delta \geq 0$ $\quad \Delta = a_1^2 - 4a_0a_2$

Assume

$$a_2 z^2 + a_1 z + a_0 = 0 \qquad a_2 > 0$$

$$\Updownarrow$$

$$\boxed{z^2 + \frac{a_1}{\boxed{a_2}} z + \frac{a_0}{\boxed{a_2}} = 0}$$

$$0 = \left(z + \frac{a_1}{2a_2}\right)^2 + \gamma = z^2 + \frac{a_1}{a_2} z + \frac{a_0}{a_2} \overset{?}{=} 0$$

$$\|$$

$$\cancel{z^2} + \cancel{\frac{a_1}{a_2} z} + \frac{a_1^2}{4a_2^2} + \gamma = \cancel{z^2} + \cancel{\frac{a_1}{a_2} z} + \frac{a_0}{a_2}$$

$$\gamma = \frac{a_0}{a_2} - \frac{a_1^2}{4a_2^2}$$

$$\left(z + \frac{a_1}{2a_2}\right)^2 = -\gamma$$

$$\left(z + \frac{a_1}{2a_2}\right)^2 = \underbrace{\frac{a_1^2}{4a_2} - \frac{a_0}{a_2}}_{\text{if this is } \geq 0}$$

$$z + \frac{a_1}{2a_2} = \pm\sqrt{\left(\frac{a_1^2}{4a_2} - \frac{a_0}{a_2}\right)}$$

Theorem: ("Fundamental Theorem of Algebra")

Given a complex polynomial $P(z)$ of n-th degree

$$P(z) = \alpha_n z^n + \dots + \alpha_1 z + \alpha_0$$

there exists $N \leq n$ such that and $z_1, \dots, z_N \in \mathbb{C}$

$$P(z) = \alpha_n (z - z_1)^{c_1} (z - z_2)^{c_2} \cdots (z_1 - z_N)^{c_N}$$

$$c_1 + c_2 + \dots - c_N = n$$

In particular $z_1, \dots, z_N$ are all the solutions of $P(z) = 0$

other words

$z_1 \cdots z_N$ are the roots $P(\bar{z})$