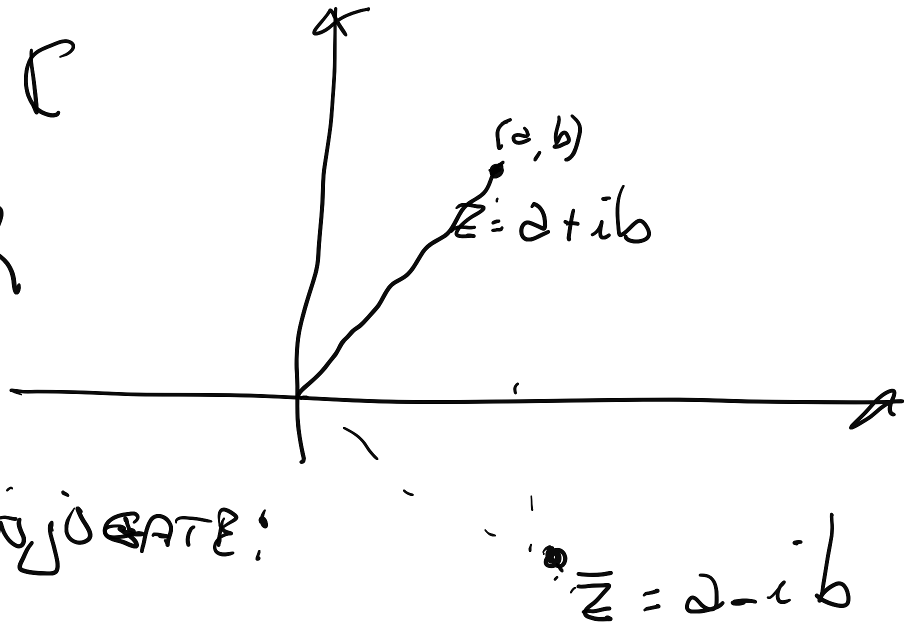
$z = a + ib \in \mathbb{C}$

$\Updownarrow$

$(a,b) \in \mathbb{R}\times\mathbb{R}$



$|z| = \sqrt{a^2+b^2}$

COMPLEX CONJUGATE:

$\bar z = a - ib$

$z\bar z = a^2+b^2$

$|z| = \sqrt{z\cdot\bar z}$

i) $\underbrace{|z|}_{=\sqrt{a^2+b^2}} = 0 \iff z = 0$

ii) $|zw| = |z|\,|w| \qquad \forall z, w \in \mathbb{C}$

iii) $|z+w| \le |z| + |w| \qquad$ " "

$\Updownarrow$

$\boxed{|z+w|^2 \le (|z|+|w|)^2}$

$\boxed{\begin{array}{l} z = a+ib \\ w = c+id \end{array}}$

$|z+w|^2 = (a+c)^2 + (b+d)^2 =$

$$= a^2 + c^2 + 2ac + b^2 + d^2 + 2bd =$$

$$= (a^2 + b^2) + (c^2 + d^2) + 2(ac + bd)$$

$$|z|^2 + |w|^2 + 2(ac + bd) \leq$$

If I am able to show that $\boxed{ac + bd \leq |z| \cdot |w|}$ ✱

$$\leq |z|^2 + |w|^2 + 2|z| \cdot |w| = (|z| + |w|)^2$$

Let us prove $\boxed{}$:

if $ac + bd < 0$ nothing to prove

if $ac + bd \geq 0$

$$✱ \iff (ac + bd)^2 \leq |z|^2 |w|^2$$

$$\cancel{a^2c^2} + \cancel{b^2d^2} + 2acbd \leq (a^2 + b^2)(c^2 + d^2)$$

$$= \cancel{a^2c^2} + a^2d^2 + b^2c^2 + \cancel{b^2d^2}$$

$$\Updownarrow$$

$$2acbd \leq a^2d^2 + b^2c^2$$

$$2(ad)(bc) \leq a^2d^2 + b^2c^2$$

$$\Uparrow$$

$$0 \leq \underbrace{a^2d^2 + b^2c^2 - 2(ad)(bc)}_{(ad-bc)^2}$$

Exercise :

$$z^2 + 2\bar{z} = |z|^2 \qquad z \in \mathbb{C}$$

$z = x + iy$

$z^2 = (x+iy)^2 \quad \cancel{x^2} - y^2 + 2ixy + 2(x - iy) = \cancel{x^2} + y^2$

$$-2y^2 + 2ixy + 2x - 2iy =$$

$$\underline{(-2y^2 + 2x)} + i\,\underline{(2xy - 2y)} = 0$$

$$\begin{cases} -\cancel{2}y^2 + \cancel{2}x = 0 \\ \cancel{2}xy - \cancel{2}y = 0 \end{cases} \iff \begin{cases} y^2 = x \\ y(x-1) = 0 \end{cases}$$

$$\begin{cases} y^2 = x \\ y = 0 \end{cases} \qquad \text{or} \qquad \begin{cases} y^2 = x \\ x - 1 = 0 \end{cases}$$

$\underline{(0,0)}$ $\qquad\qquad\qquad\qquad$ $\Downarrow$

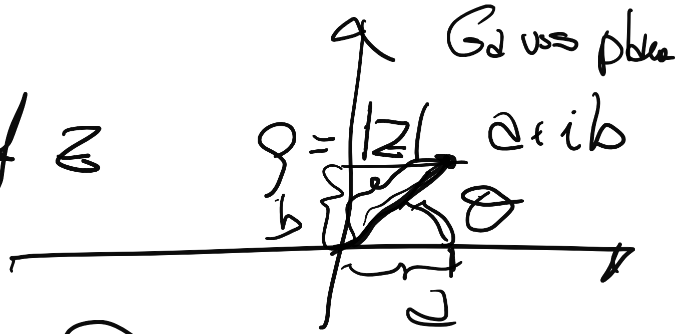
$$\begin{cases} y^2 = 1 \\ x = 1 \end{cases}$$

The solutions $(1,1)$ $(1,-1)$

$z_1 = 0$ $z_2 = 1+i$ $z_3 = 1-i$

$z = a + ib$

algebraic representation of z



$$z = \rho \cos\theta + i \rho \sin\theta =$$

$$= \rho \underbrace{\boxed{(\cos\theta + i \sin\theta)}}_{\text{unit circle}}$$

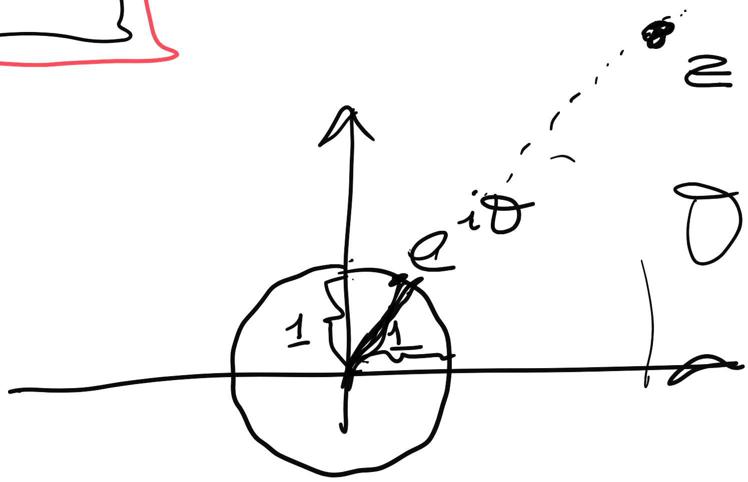
ρ = "rho" modulus

$|z|$

θ is the "argument"

$$\boxed{e^{i\theta}}$$

$$z = \rho e^{i\theta}$$



$z = \rho e^{i\theta}$ $\qquad w = r e^{i\varphi}$

$$z \cdot w = \rho r e^{i\theta} e^{i\varphi} \overset{?}{=} \rho r e^{i(\theta+\varphi)}$$

$$\parallel$$

$$\rho(\cos\theta + i\sin\theta)\cdot r(\cos\varphi + i\sin\varphi) =$$

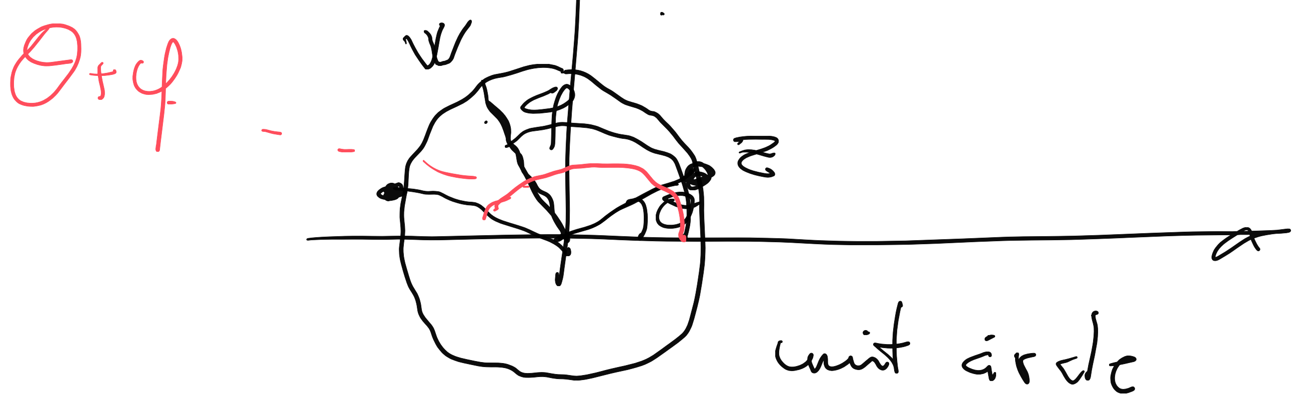
$$\rho r\left[\left(\overbrace{\cos\theta\cos\varphi - \sin\theta\sin\varphi}^{\cos(\theta+\varphi)}\right) + i\left(\overbrace{\sin\theta\cos\varphi + \cos\theta\sin\varphi}^{\sin(\theta+\varphi)}\right)\right]$$

$$= \rho \cdot r\left(\underbrace{\cos(\theta+\varphi) + i\sin(\theta+\varphi)}\right) =$$

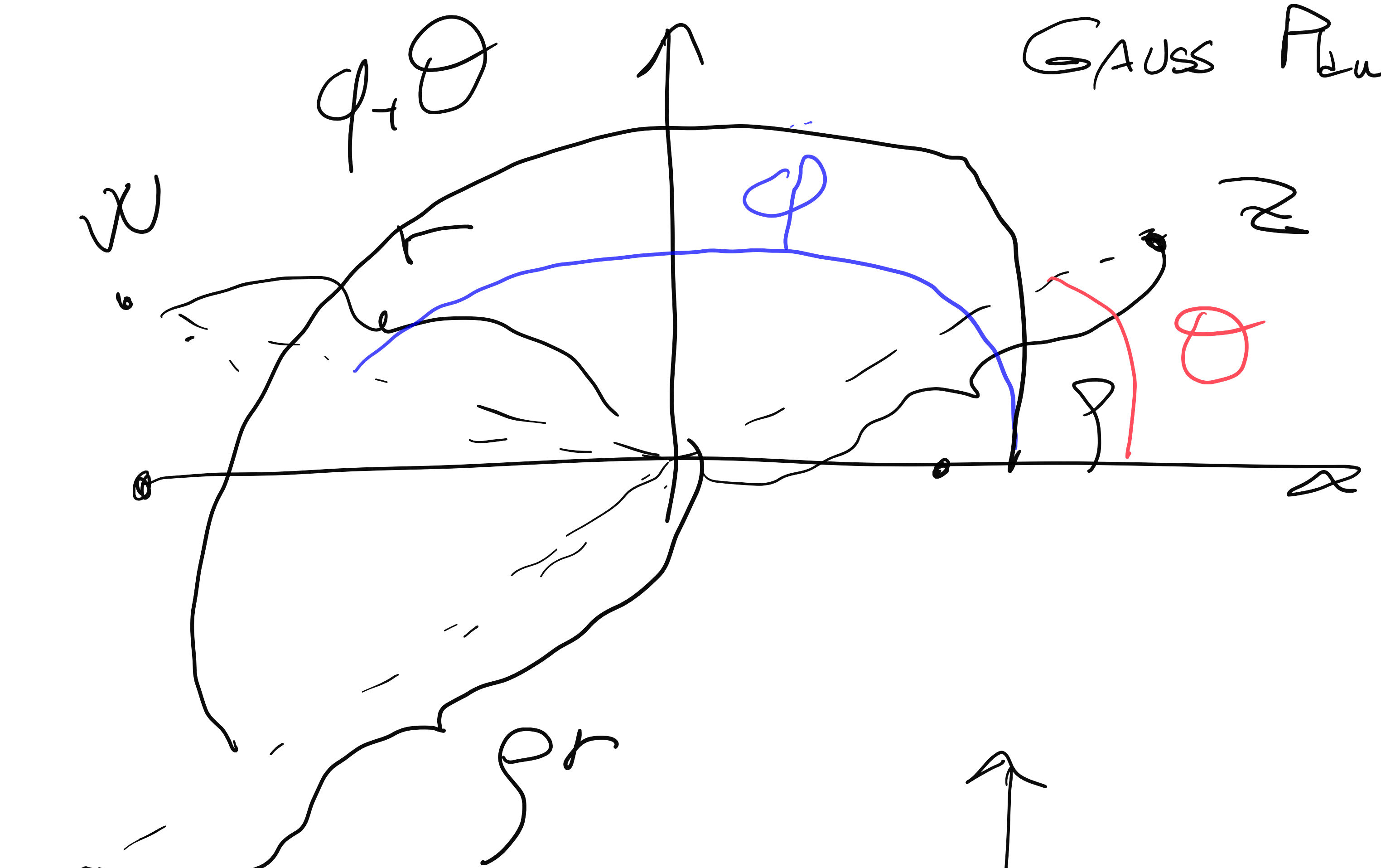
$$= \rho r e^{i(\theta+\varphi)}$$

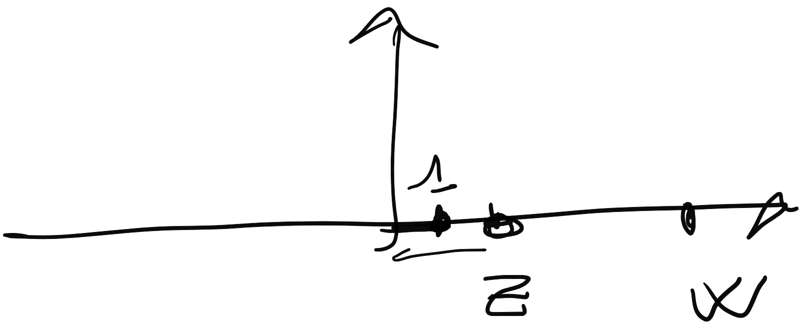
$z = \rho e^{i\theta}$ $\qquad w = r e^{i\varphi}$

$$z \cdot w = \rho r \, \underline{e^{i(\theta+\varphi)}}$$



Gauss Plane

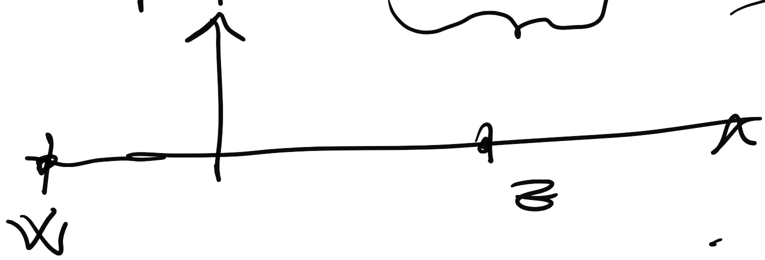




$z \in \mathbb{R} \quad w \in \mathbb{R}$

$z = |z| e^{i0} \qquad w = |z| e^{i0}$

$z \cdot w = |z||w| \underbrace{e^{i(0+0)}} = |z||w|$



$z = |z| \overbrace{e^{i0}}^{1}$

$w = |w| \underbrace{e^{i\pi}}_{-1}$

$z \cdot w = |z||w| e^{i(0+\pi)} =$

$= |z||w| e^{i(\pi)} =$

$= -|z||w|$

Problem: give $z \in \mathbb{C}$ and find the n-th roots w, that is

$$w^n = z$$

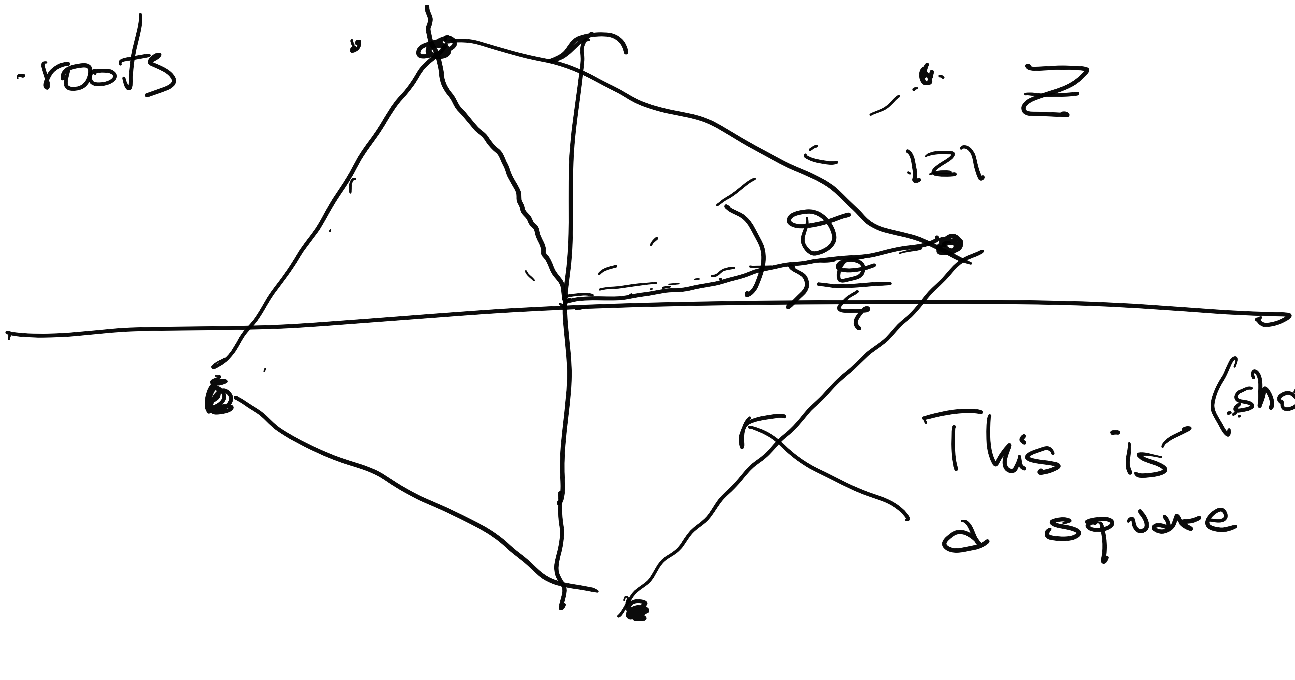
<u>Theorem</u>: Any point $z \in \mathbb{C}$ $z \neq 0$, there exist n distinct roots given by the following formula:

$$z = \rho e^{i\theta}$$

$k = 0, \ldots, n-1$

$$w_0 = |z|^{\frac{1}{n}} e^{i\frac{\theta}{n}}$$
$$w_1 = |z|^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}\right)}$$
$$w_2 = |z|^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{4\pi}{n}\right)}$$
$$w_k = |z|^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}$$
$$\vdots$$
$$w_{n-1} = |z|^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2(n-1)\pi}{n}\right)}$$

4th-roots



This is (should be) a square