Lesson 5 - 10/10/2022

Classification of 2x2 linear systems, with matrix A diagonalizable.

$$\begin{cases} \dot{z} = Az \\ z(0) = z_0 \end{cases} \qquad A \text{ diag. } 2\times 2 \qquad \varphi_t(z_0) = e^{tA} z_0$$

$\boxed{\text{I}}$ $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \neq 0$.

$\lambda_1 \to v_1 \in \mathbb{R}^2$, $\lambda_2 \to v_2 \in \mathbb{R}^2$ $\qquad$ v_1, v_2 are linearly indep.

$$z_0 = c_1 v_1 + c_2 v_2$$

$$e^{tA} z_0 = e^{tA}[c_1 v_1 + c_2 v_2] =$$

$$= c_1 \underbrace{e^{tA} v_1} + c_2 \underbrace{e^{tA} v_2}$$

$$e^{tA} v_1 = \sum_{k=0}^{+\infty} \frac{1}{k!} (tA)^k v_1 =$$

$$= \sum_{k=0}^{+\infty} \frac{1}{k!} (tA)^{k-1} \underbrace{A v_1}_{= \lambda_1 v_1} = \dots \overset{\text{repeating the argument}}{=} \underbrace{\sum_{k=0}^{+\infty} \frac{1}{k!} t^k \lambda_1^k}_{e^{\lambda_1 t}} v_1 =$$

$$= e^{\lambda_1 t} v_1 .$$

$$e^{tA} v_2 = e^{\lambda_2 t} v_2$$

$$e^{tA} z_0 = \boxed{\varphi_t(z_0) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2}$$

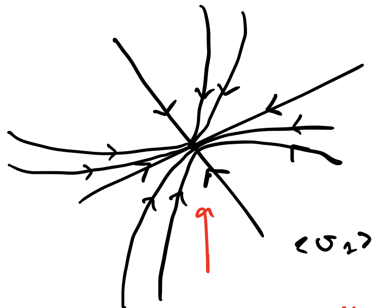
From the previous formula, we obtain that

If $c_1 = 0$ then $\varphi_t(z_0) = c_2 e^{\lambda_2 t} v_2 \to \langle v_2 \rangle$ invariant eigenspace.

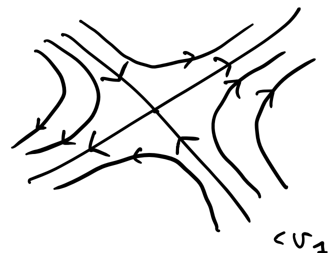
If $c_2 = 0$ then $\varphi_t(z_0) = c_1 e^{\lambda_1 t} v_1 \to \langle v_1 \rangle$ invariant eigenspace.

$\lambda_1 < \lambda_2 < 0$

$\lambda_1 < 0 < \lambda_2$

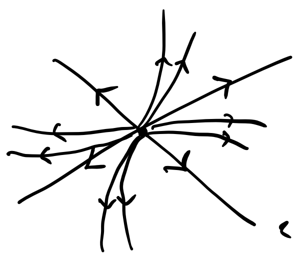


STABLE NODE



SADDLE

$0 < \lambda_2 < \lambda_1$

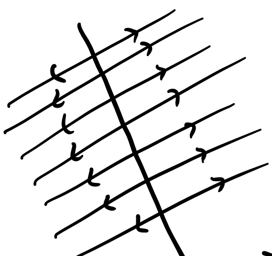


UNSTABLE NODE

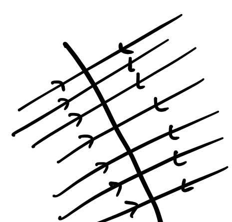
$\boxed{2}$ $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$ and $\lambda_2 = 0$.

In such a case:

$$\varphi_t(z_0) = \varphi_t(c_1 v_1 + c_2 v_2) = c_1 e^{\lambda_1 t} v_1 + c_2 v_2$$



$\lambda_1 > 0 = \lambda_2$



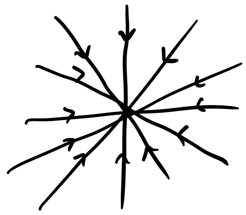
All fixed points

$\boxed{3}$ $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 = \lambda_2$.

In such a case, the eigenspace corresponding to $\lambda_1 = \lambda_2$ has dim. $= 2$. $\Rightarrow$ All vectors of $\mathbb{R}^2$ are eigenvectors for $\lambda_1 = \lambda_2$ that $Av = \lambda_1 v \quad \forall \; v \in \mathbb{R}^2$.

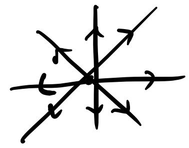
In other words,

$$\varphi_t(v) = e^{\lambda_1 t} v \quad \forall\, v \in \mathbb{R}^2.$$



$\lambda_1 = \lambda_2 < 0$

STABLE STAR NODE



$\lambda_1 = \lambda_2 > 0$

UNSTABLE STAR NODE

$\boxed{4}$ Eigenvalues are complex numbers $\alpha \pm i\beta$

$\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

From linear algebra: in such a case we can always take eigenvectors:

$$\begin{cases} v_1 = u + iw \\ v_2 = u - iw \end{cases} \qquad u, w \in \mathbb{R}^2$$

$z_0 = c_1 v_1 + c_2 v_2$

Moreover, $\varphi_t(z_0) \in \mathbb{R}^2$ if also c_1 and c_2 are of this form ($z_0 \in \mathbb{R}^2$)

$$\begin{cases} c_1 = \rho e^{i\varphi} \\ c_2 = \rho e^{-i\varphi} \end{cases}$$

Now we can write the solution starting from z_0 at time t.

$$\varphi_t(z_0) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 =$$

$$= \underbrace{\rho e^{i\varphi}}_{= c_1} e^{\overbrace{t\alpha + it\beta}^{}_{}} \underbrace{(u + iw)}_{= v_1} +$$

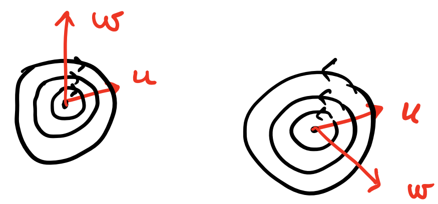
$t\alpha + it\beta = \lambda_1 t$

$$+ \underbrace{\rho e^{-i\varphi}}_{= c_2} e^{\underbrace{t\alpha - it\beta}_{= \lambda_2 t}} \underbrace{(u - iw)}_{= v_2} =$$

$$= \dots = 2\rho e^{t\alpha}\left[\cos(\varphi + t\beta)u - \sin(\varphi + t\beta)w\right]$$

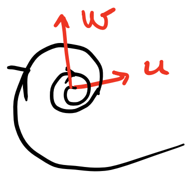
We need to use Euler's formula

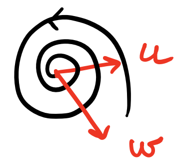
→ $\boxed{\alpha = 0}$ → Periodic motion



CENTER

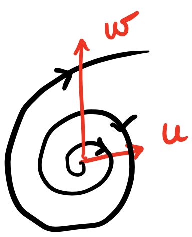
→ $\boxed{\alpha < 0}$

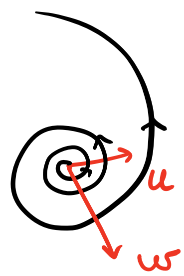




STABLE SPIRAL

→ $\boxed{\alpha > 0}$





UNSTABLE SPIRAL

EXPLICIT COMPUTATION

(NO important, clearly)

$$\varphi_t(z_0) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 =$$

$$= \rho e^{i\varphi} e^{t\alpha + it\beta}(u + iw) + \rho e^{-i\varphi} e^{t\alpha - it\beta}(u - iw) =$$

$$= \rho e^{i\varphi} e^{t\alpha} e^{it\beta}(u + iw) + \rho e^{-i\varphi} e^{t\alpha} e^{-it\beta}(u - iw) =$$

$$= \rho e^{t\alpha + i(\varphi + t\beta)}(u + iw) + \rho e^{t\alpha} e^{-i(\varphi + t\beta)}(u - iw) =$$

$$= \rho e^{t\alpha}\left[\cos(\varphi+t\beta)+i\sin(\varphi+t\beta)\right](u+iw)+$$

$\downarrow$

Euler's formula

$$+\rho e^{t\alpha}\left[\cos(-\varphi-t\beta)+i\sin(-\varphi-t\beta)\right](u-iw)$$

$$=\rho e^{t\alpha}\left[\cos(\varphi+t\beta)+i\sin(\varphi+t\beta)\right](u+iw)+$$

$$+\rho e^{t\alpha}\left[\cos(\varphi+t\beta)-i\sin(\varphi+t\beta)\right](u-iw)$$

$$=\rho e^{t\alpha}\left[\cos(\varphi+t\beta)u+i\cancel{\cos(\varphi+t\beta)w}+\right.$$

$$+i\cancel{\sin(\varphi+t\beta)}u-\sin(\varphi+t\beta)w+$$

$$+\cos(\varphi+t\beta)u-\cancel{\cos(\varphi+t\beta)iw}-i\cancel{\sin(\varphi+t\beta)}u-$$

$$\left.-\sin(\varphi+t\beta)w\right]=$$

it's $\ominus$!!

$$=\rho e^{t\alpha}\left[2\cos(\varphi+t\beta)u-2\sin(\varphi+t\beta)w\right]$$

$$=2\rho e^{t\alpha}\left[\cos(\varphi+t\beta)u-\sin(\varphi+t\beta)w\right]$$

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Ex Solve the initial value problem $\begin{cases}\dot{x}=x+y\\ \dot{y}=4x-2y\end{cases}$

with $(x_0,y_0)=(2,-3)$.

Sol $A=\begin{pmatrix}1 & 1\\ 4 & -2\end{pmatrix}$

$\det[A-\lambda \mathbb{1}]=0 \iff \lambda^2+\lambda-6=0$

Hence $\lambda_1=2$ and $\lambda_2=-3$

$\downarrow$ around $(0,0)$, the unique fixed point, we have a saddle. $(\lambda_1>0,\ \lambda_2<0)$

v_2 = eigenvector corresp. to $\lambda_2 < 0 \to v_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ -3

v_1 = eigenvector corresp. to $\lambda_1 > 0 \to v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 2

The corresp. flow is

$$\varphi_t(z_0) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

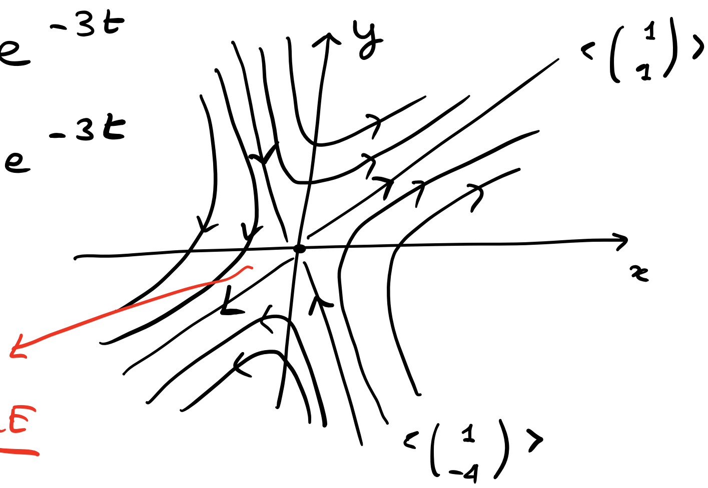
we need to fix the initial datum to $\underbrace{(2,-3)}_{= z_0}$

$$\varphi_0(z_0) = \underset{\substack{\| \\ (2,-3)}}{z_0} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\Longleftrightarrow \begin{cases} c_1 + c_2 = 2 \\ c_1 - 4c_2 = -3 \end{cases} \Longleftrightarrow \begin{cases} c_1 = 1 \\ c_2 = 1 \end{cases}$$

with this initial datum we have

$$\begin{cases} x(t) = e^{2t} + e^{-3t} \\ y(t) = e^{2t} - 4e^{-3t} \end{cases}$$



BIFURCATION DIAGRAM FOR $\dot{z} = Az$, A 2×2 DIAGONALIZ.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

In order to determine eigenvalues, we study

$\chi_A(\lambda) = \det(A - \lambda \mathbb{1}) =$

$$= \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = ad - a\lambda - d\lambda + \lambda^2 - cb$$

$$= \lambda^2 - \underbrace{(a+d)}_{= \operatorname{tr} A}\lambda + \underbrace{(ad - bc)}_{\det A} = \lambda^2 - \operatorname{tr} A\, \lambda + \det A$$

$$\Delta = (\operatorname{tr} A)^2 - 4 \det A .$$

We need to take into account three diff. cases:

① $\Delta > 0 \iff (\operatorname{tr} A)^2 - 4\det A > 0$: 2 real eigenvalues, distinct.

② $\Delta = 0 \iff (\operatorname{tr} A)^2 - 4\det A = 0$: 2 real eigenvalues, coincident

③ $\Delta < 0 \iff (\operatorname{tr} A)^2 - 4\det A < 0$: 2 complex eigenvalues, $\alpha \pm i\beta$

Moreover:

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{\Delta}}{2} = \frac{\operatorname{tr} A \pm \sqrt{\Delta}}{2}$$

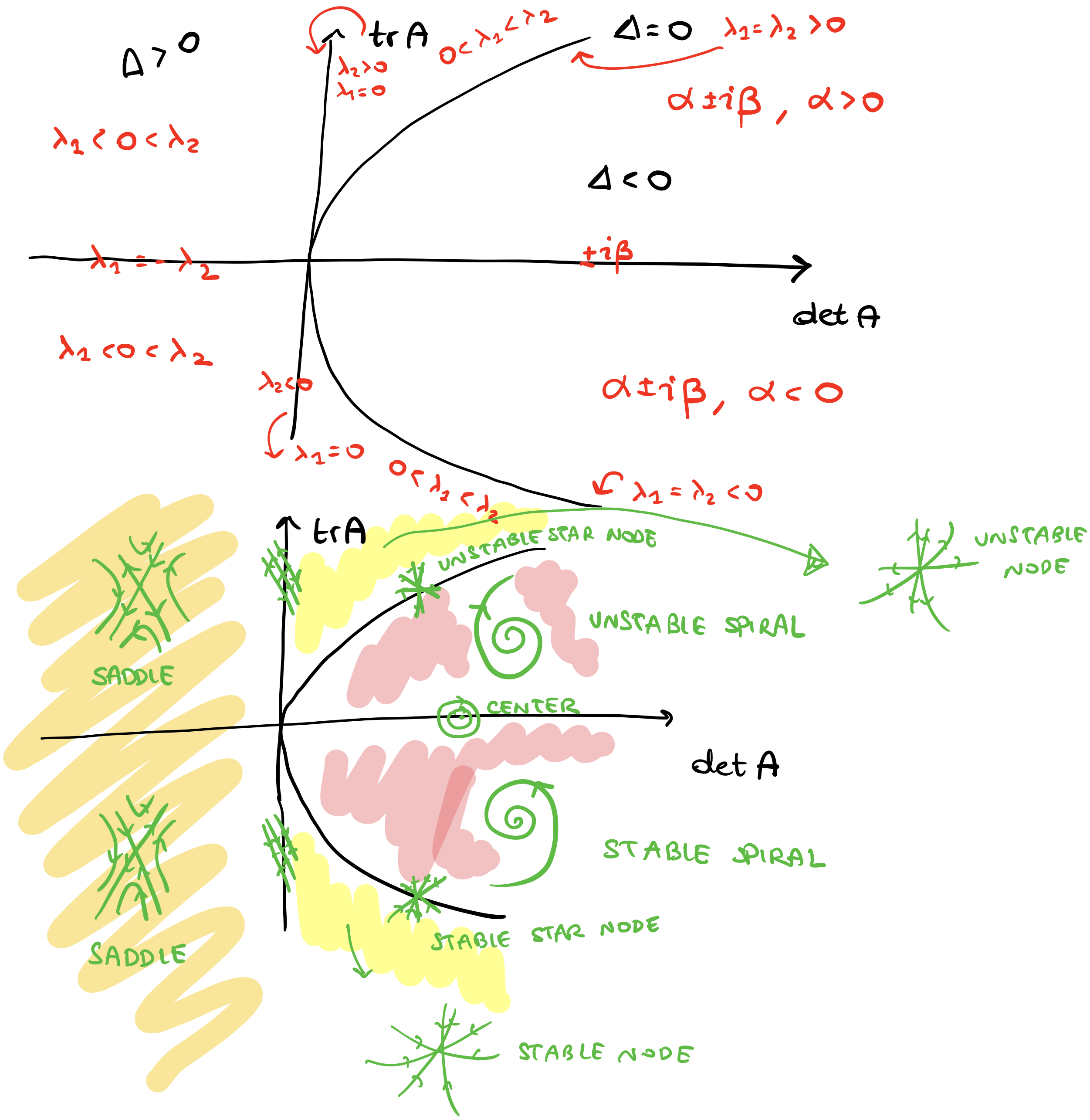
$$\Rightarrow \lambda_1 + \lambda_2 = \operatorname{tr} A \quad \text{and} \quad \operatorname{tr} A = 2\operatorname{Re}(\alpha \pm i\beta) = 2\alpha$$

$\downarrow$ complex eigenv.

$$\lambda_1 \cdot \lambda_2 = \frac{\operatorname{tr} A + \sqrt{\Delta}}{2} \cdot \frac{\operatorname{tr} A - \sqrt{\Delta}}{2} = \frac{(\operatorname{tr} A)^2 - \Delta}{4} =$$

expression for Δ

$$= \frac{\cancel{(\operatorname{tr} A)^2} - \cancel{(\operatorname{tr} A)^2} + 4 \det A}{4} = \det A$$



$\Delta > 0$
tr A
$0 < \lambda_1 < \lambda_2$
$\Delta = 0$
$\lambda_1 = \lambda_2 > 0$
$\lambda_2 > 0$
$\lambda_1 = 0$
$\lambda_1 < 0 < \lambda_2$
$\alpha \pm i\beta$, $\alpha > 0$
$\Delta < 0$
$\lambda_1 = -\lambda_2$
$\pm i\beta$
det A
$\lambda_1 < 0 < \lambda_2$
$\lambda_2 < 0$
$\alpha \pm i\beta$, $\alpha < 0$
$\lambda_1 = 0$
$0 < \lambda_1 < \lambda_2$
$\lambda_1 = \lambda_2 < 0$
tr A
UNSTABLE STAR NODE
UNSTABLE NODE
SADDLE
UNSTABLE SPIRAL
CENTER
det A
SADDLE
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