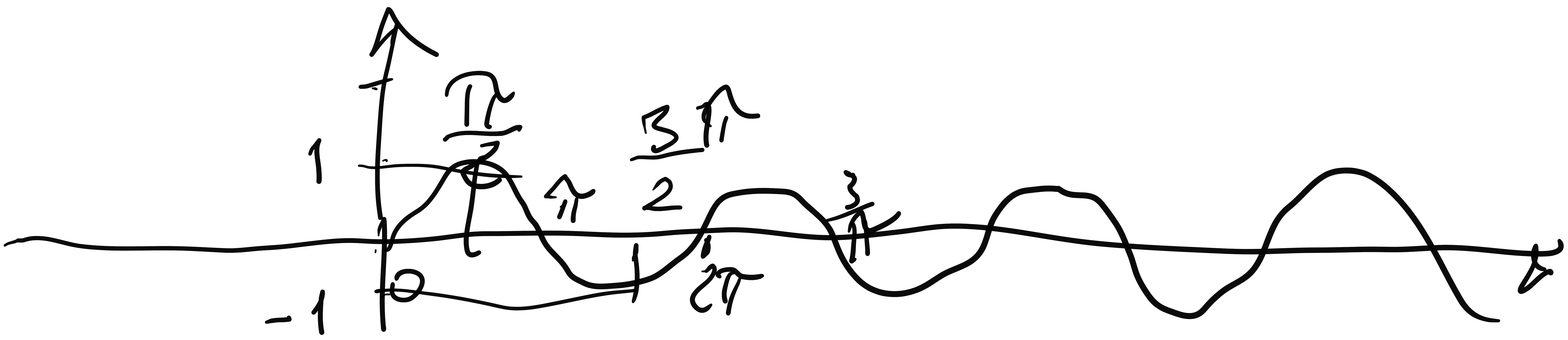
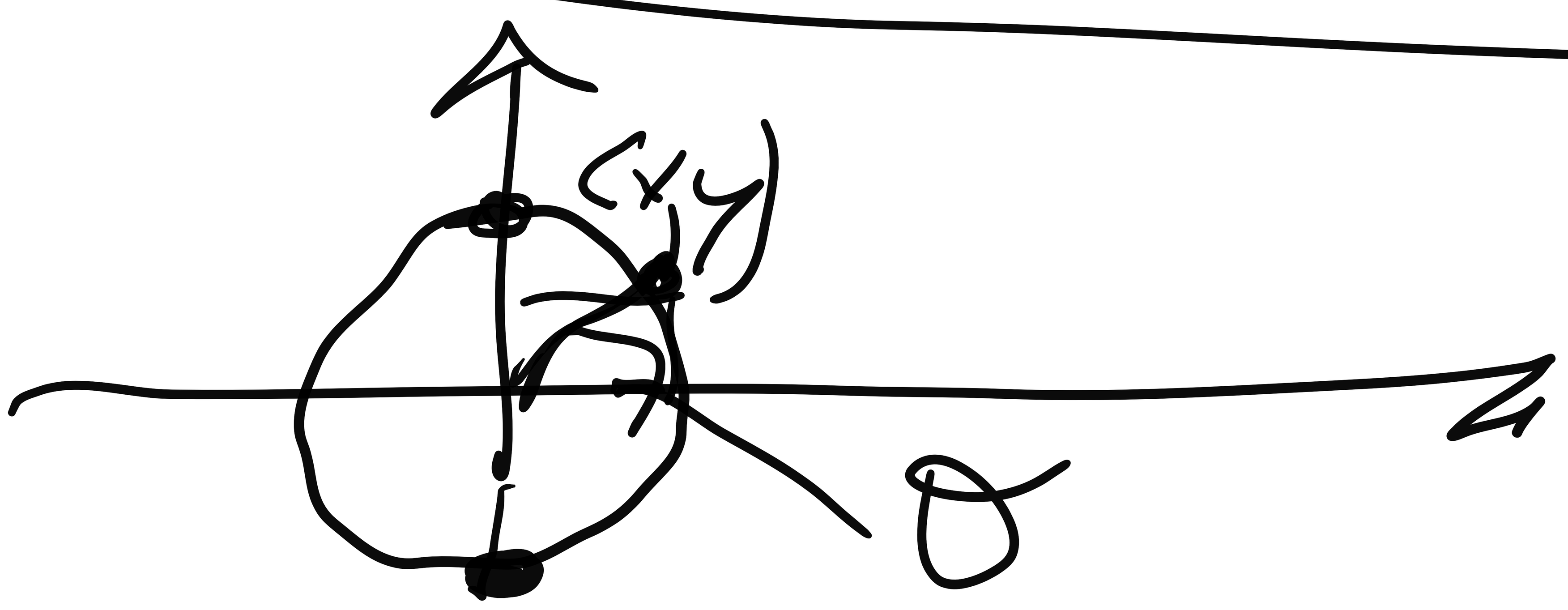
$\sin : \mathbb{R} \longrightarrow \mathbb{R}$ non surjective




$x =: \cos\theta$

$y =: \sin\theta$

$\sin \; \mathbb{R} \longrightarrow [-1, 1]$ surjective

$f : A \longrightarrow B$ $\quad P \Longrightarrow Q$

injectivity: $a_1 \neq a_2 \Longrightarrow f(a_1) \neq f(a_2)$

$\Updownarrow$

non $Q \Longrightarrow$ non $P$

$f(a_1) = f(a_2) \Longrightarrow a_1 = a_2$

injectivity of sine ?

$$\sin(x_1) = \sin(x_2)$$

$$? \quad \Downarrow \quad ?$$

$$x_1 = x_2$$

NO  if $x_2 = x_1 + 4\pi$

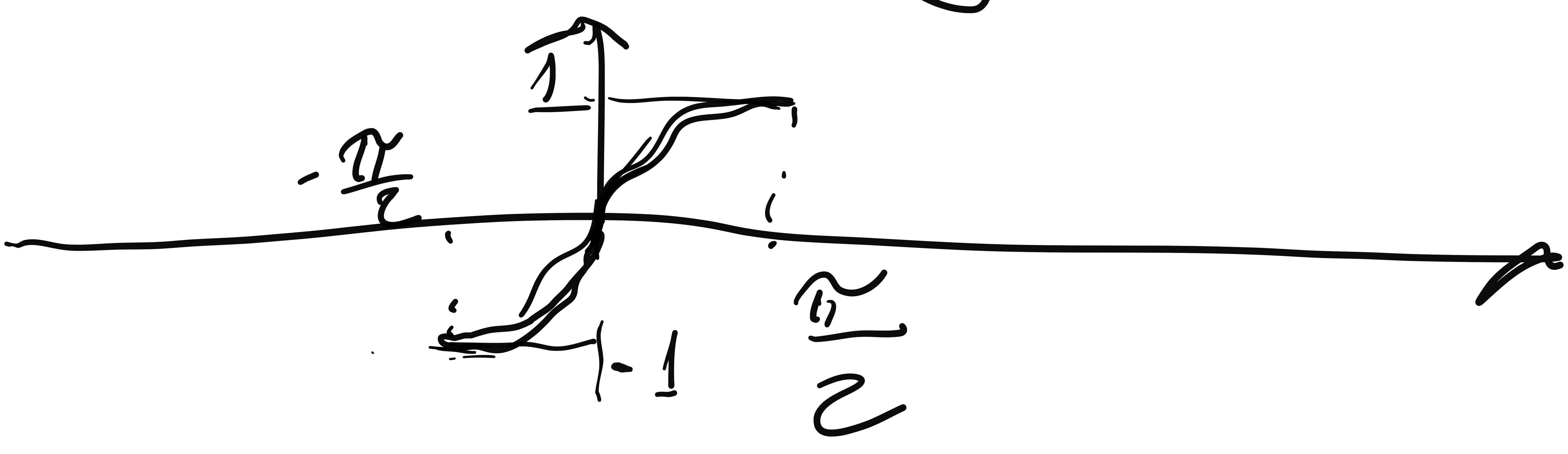
---

To get injectivity

restrict to $I = [\bar{x}, \bar{x} + \pi]$

I can choose suitable $\bar{x} = -\frac{\pi}{2}$

$$I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



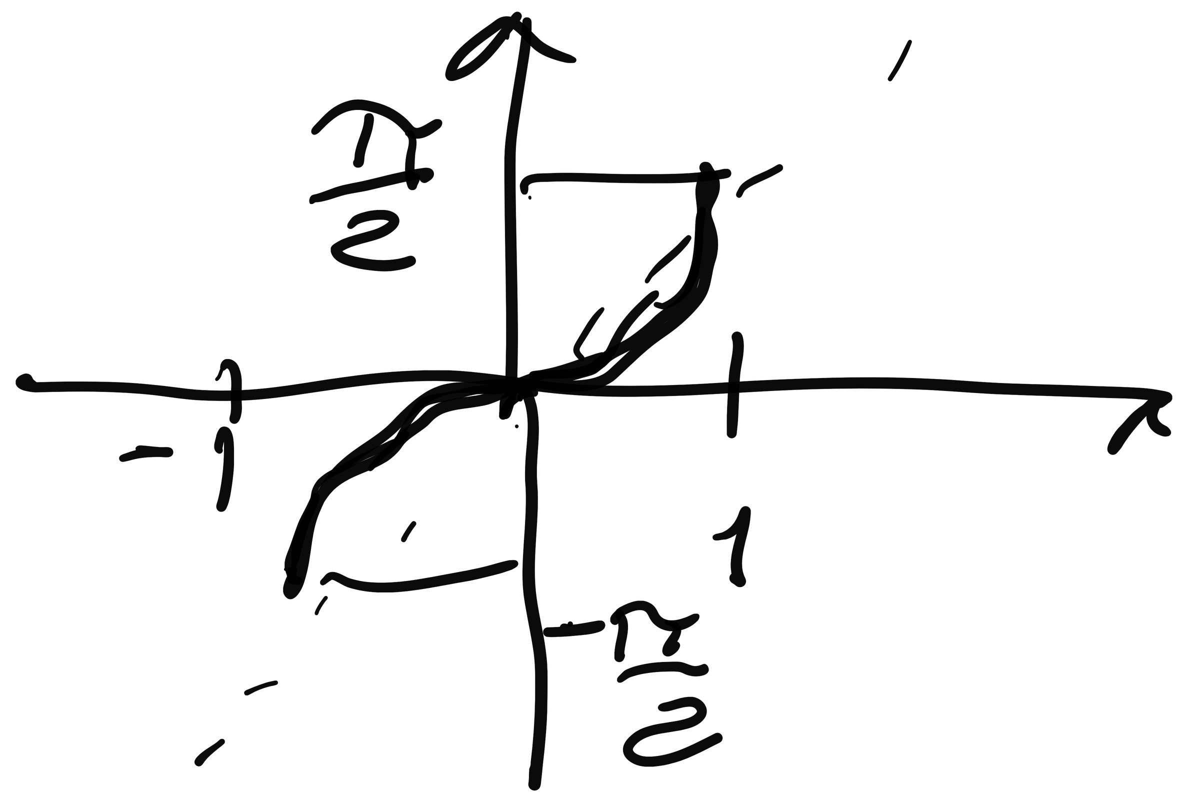

$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$

this function is

bijective
Its inverse is called arcsine

$\arcsin : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

arc sin

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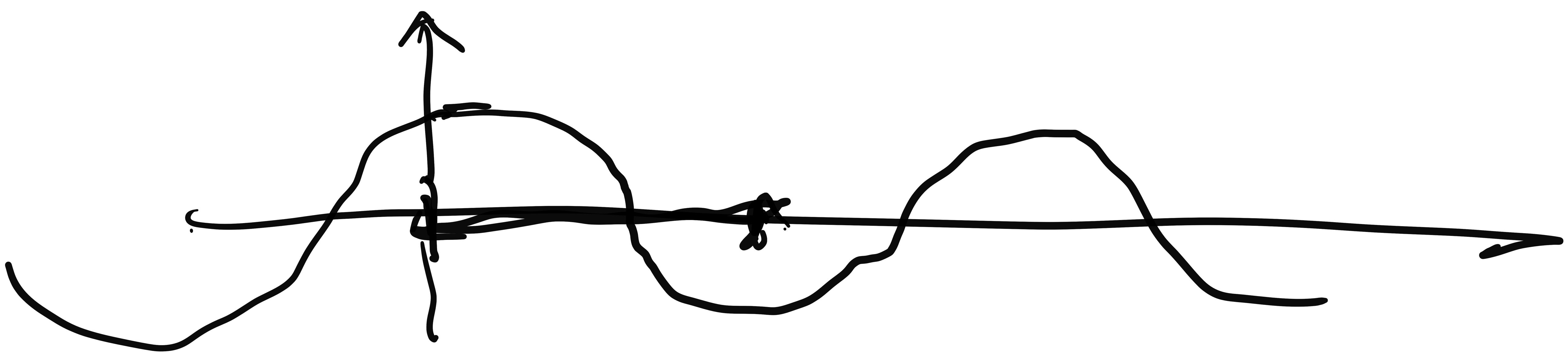
$\cos : \mathbb{R} \longrightarrow \mathbb{R}$



$\cos [0, \pi] \longrightarrow [-1, 1]$

: this is bijective $\iff$ invertible

$\underline{\arccos} : [-1, 1] \longrightarrow [0, \pi]$

(decreasing)

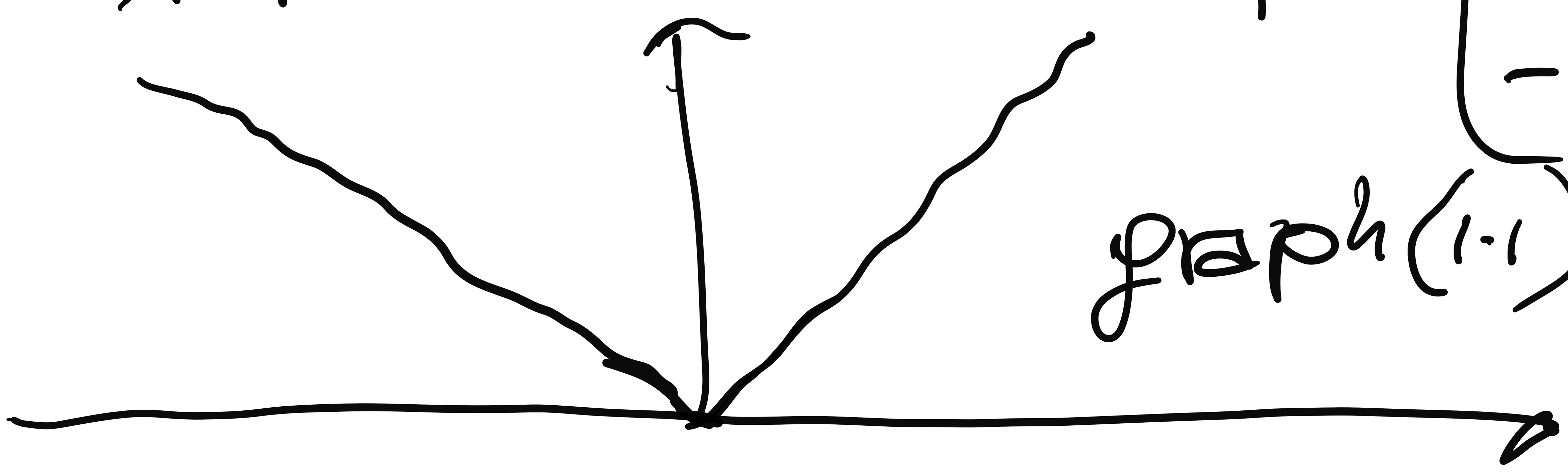
---

$\sin(\theta_1 + \theta_2) = \ldots$

$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$

# The "modulus" function (absolute value)

$|\cdot| \quad x \longmapsto |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$



graph($|\cdot|$)

$|xy| = |x|\,|y|$

$x \geq 0 \quad y \geq 0 \Rightarrow xy \geq 0 \Rightarrow \boxed{|xy| = xy = |x|\,|y|}$

$x = |x| \quad y = |y|$

$x \geq 0 \quad y \leq 0 \Rightarrow xy \leq 0 \quad \boxed{|xy| = -xy}$

$\Updownarrow \qquad \Updownarrow$

$|x| = x \qquad |y| = -y \qquad \boxed{|x|\,|y| = -xy}$

q.e.d.

Triangular property of $|\cdot|$:

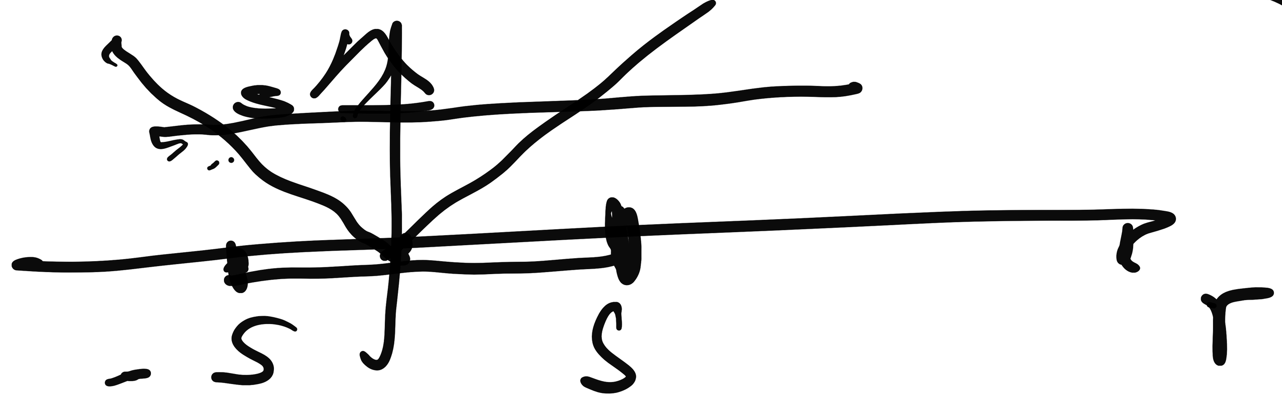
$$\boxed{|x+y| \leq |x| + |y|}$$

$s \geq 0$

$|r| \leq s$

$\Updownarrow$

$-s \leq r \leq s$




$-|x| - |y| \leq x+y \leq |x| + |y|$

yes, because

$\begin{cases} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{cases}$

add these

Exercise Solve

$$2^{|x-1|} < 4^{-x}$$

$$\boxed{2^{|x-1|}} < (2^2)^{-x} = \boxed{(2)^{-2x}}$$

$$\Updownarrow$$

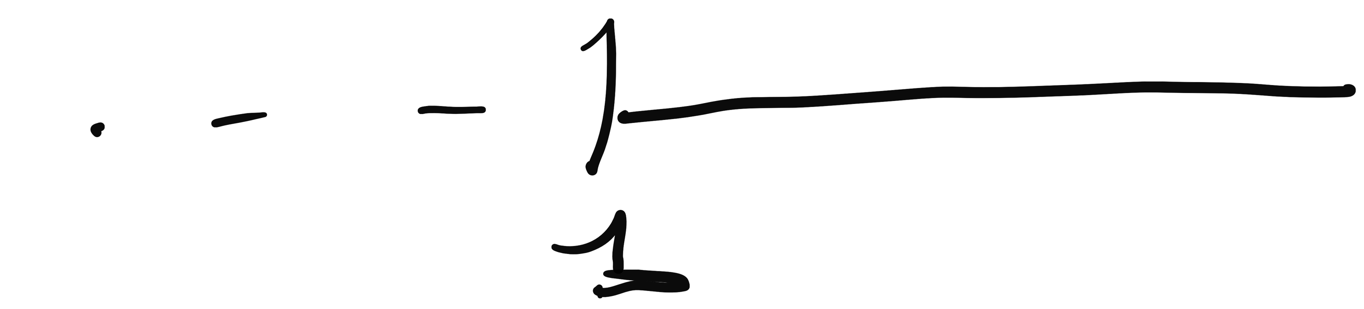
$$\boxed{(*) \quad |x-1| < -2x}$$

I case
$$\begin{cases} \boxed{x-1 \geq 0} \\ \boxed{x-1 < -2x} \end{cases}$$

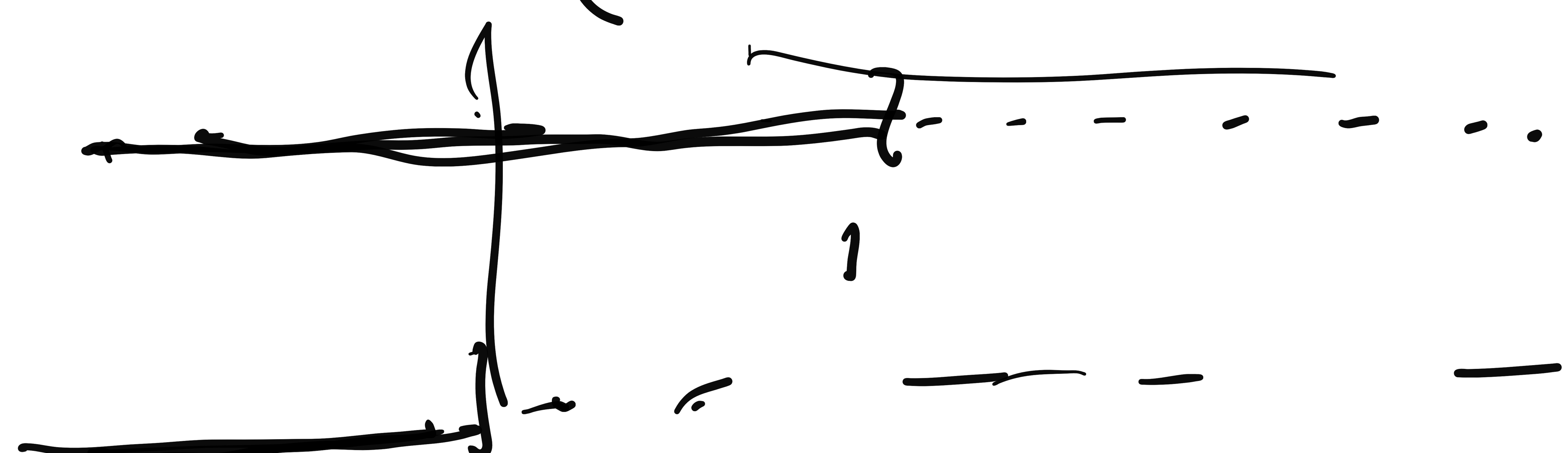
$\boxed{\text{or}}$

II case
$$\begin{cases} \boxed{x-1 < 0} \\ \boxed{1-x < -2x} \end{cases}$$

(*)
$$\begin{cases} x \geq 1 \\ x < \frac{1}{3} \end{cases} \qquad \begin{cases} x < 1 \\ x < -1 \end{cases}$$



no solution



$-1$

$$\boxed{x < -1}$$

A polinomial is

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

A polinomial equation is

$$(**) \quad a_n x^n + a_{n-1} x^{n-1} + a_2 x^2 + a_1 x + a_0 = 0$$

In general, are there solutions of (**)?

$\boxed{\text{No}}$

$x^2 + 1 = 0$ has no real solutions.

$$\Updownarrow$$

$$x^2 = -1$$

Introduce $i$ $\quad i^2 = -1$

$$\underbrace{(13 + 5i)} + (\pi + 9i) =$$

$$(13 + \pi) + 14i$$

$$(6 + 0i) + (3 + 0i) = (6 + 3) + i(0 + 0)$$

$$= 9 + i \cdot 0 = 9$$

We are considering objects like

$z = x + iy$ $\qquad \mathbb{R} \ni x :=$ "real part"

$\qquad \mathbb{R} \ni y :=$ "imaginary part"

We will call these new objects "complex numbers"

$$\mathbb{C} = \{ z = x + iy \quad (x, y) \in \mathbb{R} \times \mathbb{R} \}$$

**sum** $z_1 + z_2 = \underset{z_1}{(x_1 + i y_1)} + \underset{z_2}{(x_2 + i y_2)} = (x_1+x_2) + i(y_1+y_2)$

check 1) associative property

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ $\forall z_1, z_2, z_3 \in \mathbb{C}$

2) commutative property

$z_1 + z_2 = z_2 + z_1$ $\forall z_1 z_2 \in \mathbb{C}$

3) neutral element

$0 = 0 + i \cdot 0$

4) Existence of a opposite of $z = x + iy$

$-z = -x - iy$

---

Product

$z_1 = x_1 + i y_1$

$z_2 = x_2 + i y_2$

$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

$z_1 \cdot z_2 = (x_1 + i y_1)(x_2 + i y_2) = x_1 x_2 + i x_1 y_2 + i y_1 x_2 - y_1 y_2 =$

$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

Check associativity $z_1(z_2 z_3) = (z_1 \cdot z_2) \cdot z_3$

commutativity $z_1 z_2 = z_2 z_1$

for the product

Neutral number is 1

$\mathbb{1} = 1 + i0$

$z \cdot \mathbb{1} = (x+iy) \cdot (1+i0) = x+iy = z$

$z = x+iy$

---

$z_1 = x_1 + iy_1 \quad z_2 = x_2 + iy_2$

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} \cdot \frac{x_2-iy_2}{x_2-iy_2} =$$

$$\frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)}{x_2^2 - (iy_2)^2} =$$

$$\frac{(x_1x_2+y_1y_2) + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}$$

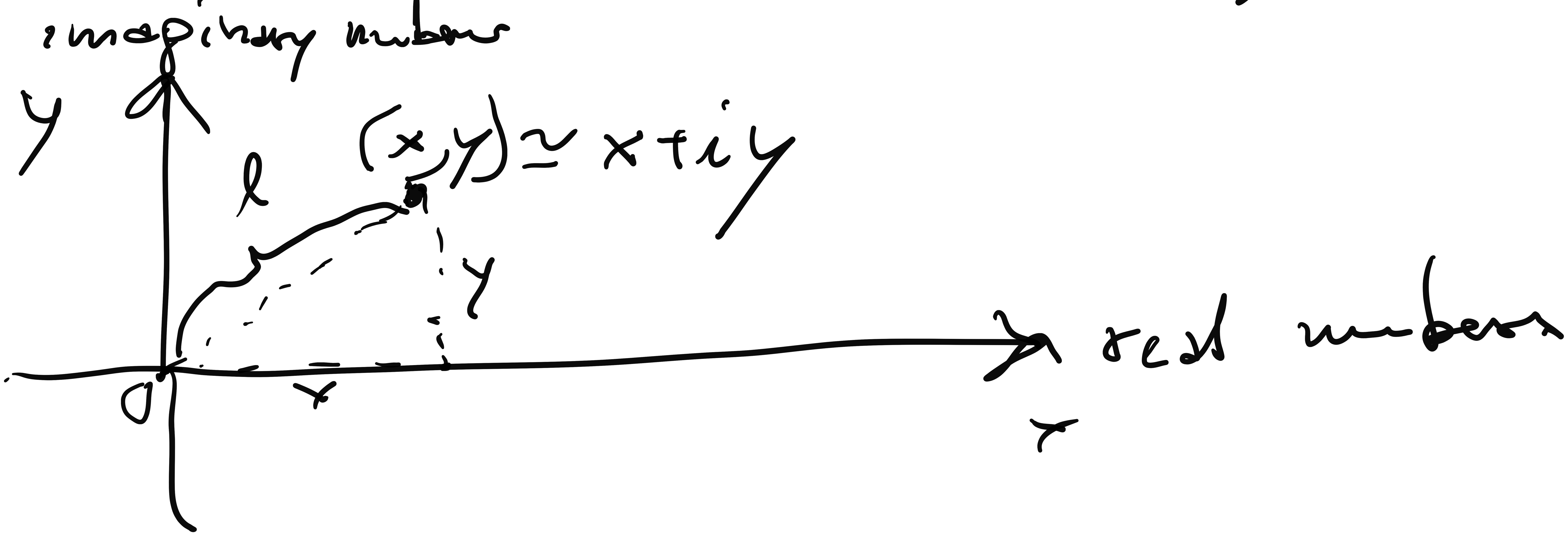
Definition: $z \in \mathbb{C}$  $z = x + iy$

The "complex conjugate" of $z$ is

$$\bar{z} := x - iy$$

Important

$$z \cdot \bar{z} = (x+iy)(x-iy) = x^2 + y^2$$




$$l^2 = x^2 + y^2 \quad l = \sqrt{x^2+y^2}$$ Pithagoras

Def

$$\boxed{|z|} = \sqrt{x^2+y^2} = \boxed{\sqrt{z \cdot \bar{z}}}$$

$$z^2 + 2\bar{z} = |z|^2$$

$$S := \left\{ \left| \frac{z+1}{z} \right| \geq 1 \right\}$$