MATH4U - 06/10/2022

Friday, 30 September 2022 09:10

1) INTRODUCTION MATH4U
2) QUESTIONS ABOUT "REVIEW EXERCISES"
3) INJECTIVE / SURJECTIVE FUNCTIONS
4) SUP/INF - MAX/MIN
5) FUNCTION DOMAINS
(6) INDUCTION)

Heuristic : the maths of solving problems

4 steps =
- have an idea
- define a plan
- carry out the plan / do the calculations
- check the solution

QUADRATIC FORMULA

$ax^2+bx+c=0$

$$x_{12} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

b is even : $x_{12} = \dfrac{-\frac{b}{2} \pm \sqrt{\frac{b^2}{4}-ac}}{a}$

INJECTIVITY / SURJECTIVITY

1) $f: \mathbb{Z} \to \mathbb{N}$, $z \mapsto f(z) = 3z^2+1$

- counterexample : z_1, z_2 $z_1 \neq z_2$ st $f(z_1) = f(z_2)$

  $z_{12} = \pm 1$ $f(z_1) = 3\cdot 1+1 = 4 = f(z_2) = 3\cdot(-1)^2+1 = 4$

- c.e : $0 \in \mathbb{N}$ $\exists z \in \mathbb{Z}$ s.t. $f(z)=0$ NO

$3z^2+1=0 \qquad z^2=-\frac{1}{3} \Rightarrow z \notin \mathbb{Z}$ NO □

2) $f: \mathbb{N} \to \mathbb{Z}, \quad n \mapsto f(n)=2n-4$

- Dem. $f(n_1)=f(n_2) \Rightarrow n_1=n_2 \qquad 2n_1-\cancel{4}=2n_2-\cancel{4} \qquad 2n_1=2n_2 \quad n_1=n_2$ SI
- Dem. $\nexists\, n \in \mathbb{N}$ s.t. $f(n)=1$

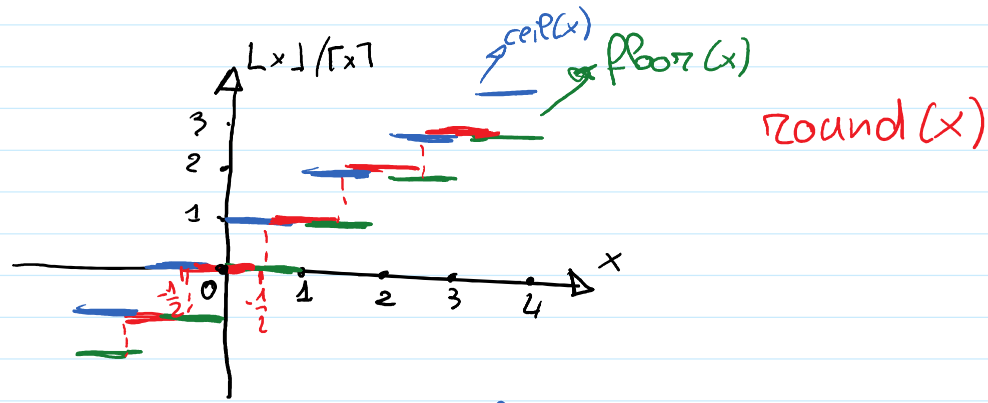
  suppose $n^* \in \mathbb{N}$ s.t. $f(n^*)=1 \qquad 2n^*-4=1 \qquad 2n^*=5$

  $n^*=\frac{5}{2} \notin \mathbb{N}$ NO

3) $f: \mathbb{R} \to \mathbb{Z}, \quad r \mapsto f(r)=\lfloor r \rfloor \to$

$\lfloor x \rfloor$ floor

$\lceil x \rceil$ ceil



- c.e. π and $3.03 \qquad f(\pi)=f(3.03)=3$ NO
- Dem. $\forall z \in \mathbb{Z} \ \exists r \in \mathbb{R}$ s.t. $f(r)=z \to r=z$ YES

4) $f: \mathbb{Z} \to \mathbb{Z}, \quad z \mapsto f(z)=z-5$

- Dem. $f(z_1)=f(z_2) \Rightarrow z_1=z_2 \qquad z_1-\cancel{5}=z_2-\cancel{5} \qquad z_1=z_2$ YES
- Dem. $\forall z \in \mathbb{Z} \ \exists z_p \in \mathbb{Z}$ s.t. $f(z_p)=z \qquad z_p-5=z \qquad z_p=z+5$ YES

CALCULATION OF SUP/INF MAX/MIN

1) $\{n : n \in \mathbb{N}^+\} \cup \{\frac{2}{n^3} : n \in \mathbb{N}^+\} \to$ $\mathbb{N} \setminus \{0\}$

$\mathbb{Z}^+$ positive integers

1) $\{n : n \in \mathbb{N}^+\} \cup \{\frac{2}{n^3} : n \in \mathbb{N}^+\}$

$\mathbb{Z}^+$ positive integers

$\mathbb{Z}^+_0$ non-negative integers

$\mathbb{Z}^+_{\geq}$

- no supremum / maximum
- $\inf(A) = 0$

1) $\frac{2}{n^3} \geq 0$ $\quad \frac{2}{n^3} > 0 \rightarrow 0 \notin A$

2) $\forall \varepsilon > 0 \ \exists \bar{a} \in A$ s.t. $\inf(A) \leq \bar{a} \leq \inf(A) + \varepsilon$

$0 \leq \bar{a} \leq 0 + \varepsilon \qquad 0 \leq \underbrace{\frac{2}{n^3} \leq \varepsilon} \qquad \varepsilon n^3 \geq 2 \qquad n \geq \sqrt[3]{\frac{2}{\varepsilon}} \in A$

$\square$

2) $\left\{\frac{n^2-1}{n^2} : n \in \mathbb{N}^+\right\}$

- $\frac{n^2-1}{n^2} = 1 - \frac{1}{n^2}$

$\sup(A) = 1$

1) $1 \geq a \ \forall a \in A \qquad \cancel{1} \geq \cancel{1} - \frac{1}{n^2} \qquad \frac{1}{n^2} \geq 0$

2) $\forall \varepsilon > 0 \ \exists \bar{a} \in A$ s.t. $\underbrace{1 - \varepsilon \leq \bar{a}} \leq 1$

$\cancel{1} - \frac{1}{n^2} \geq \cancel{1} - \varepsilon \qquad \frac{1}{n^2} \leq \varepsilon \qquad n^2 \geq \frac{1}{\varepsilon} \qquad n \geq \sqrt{\frac{1}{\varepsilon}}$

$\square$

3) $\left\{x_n = 2 + (-1)^n \left(\frac{1}{n+1}\right) : n \in \mathbb{N}, x_n \in \mathbb{R}\right\}$

n even : $x_n = 2 + \frac{1}{n+1}$ because $(-1)^n = 1$

$\rightarrow$ decreasing

$x_0 = 2 + \frac{1}{n+1} = 2 + 1 = 3$ $\quad (n = 0)$

$\rightarrow 2 < x_n \leq 3$

$n \rightarrow \infty \quad 2 + \frac{1}{\infty} = 0$

n odd : $x_n = 2 - \frac{1}{n+1}$ because $(-1)^n = -1$ $\quad n = 2m+1$

$\rightarrow$ increasing $\quad 2 - \frac{1}{2} = \frac{4}{2} - \frac{1}{2} = \frac{4-1}{2} = \frac{3}{2}$

$x_1 = 2 - \frac{1}{1+1} = \frac{3}{2}$ $\quad \rightarrow \frac{3}{2} \leq x_n < 2$

$n \rightarrow \infty \quad x_n = 2$

$\Longrightarrow \frac{3}{2} \leq x_n < 2 \ \cup \ 2 < x_n \leq 3 \Longrightarrow$ $\min(A) = 3/2$, $\max(A) = 3$

5) $\left\{ \frac{1+n}{1+n^2} : n \in \mathbb{Z} \right\}$

1) increasing n, $\frac{1+n}{1+n^2}$ increases or decreases ?

$x_n > x_{n+1} \Longleftrightarrow$ decreasing

$$\frac{1+n}{1+n^2} > \frac{1+(n+1)}{1+(n+1)^2} \longrightarrow 1+n^2+2n+1 = n^2+2n+2$$

$$\frac{1+n}{1+n^2} - \frac{n+2}{n^2+2n+2} > 0$$

$$\frac{(1+n)(n^2+2n+2)-(2+n)(1+n^2)}{(1+n^2)(n^2+2n+2)} > 0$$

$\Updownarrow$

$N > 0$

$$\cancel{n^3}+\cancel{2n^2}+2n+n^2+2n+\cancel{2}-\cancel{n^3}-n-\cancel{2n^2}-\cancel{2} > 0$$

$$n^2+3n > 0$$

$D > 0$

$1+n^2 \geq 0 \quad \forall n \in \mathbb{Z}$

$n^2+2n+2 \geq 0 \quad n_{12} = -1 \pm \sqrt{\underbrace{1-2}_{<0}} \quad \forall n \in \mathbb{Z}$

sequence is decreasing for $n < -3 \vee n > 0$

$\Updownarrow$

$n \leq -4 \vee n \geq 1$

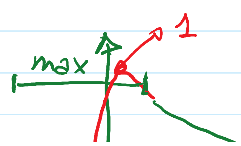
2) $\frac{1+n}{1+n^2}$ $\quad \lim_{n \to +\infty} \frac{1+n}{1+n^2} \approx \lim_{n \to +\infty} \frac{n}{n^2} = \lim_{n \to +\infty} \frac{1}{n} = 0^+$?

$\lim_{n \to -\infty} \frac{1+n}{1+n^2} \sim \lim_{n \to +\infty} \frac{n}{n^2} = 0^-$

?

min/max must be among

3) $f\{-4,-3,-2,-1,0,1\}$



$n \quad 1-4 \quad = -3$

5) $\downarrow \{-4,-3,-2,-1,0,1\}$

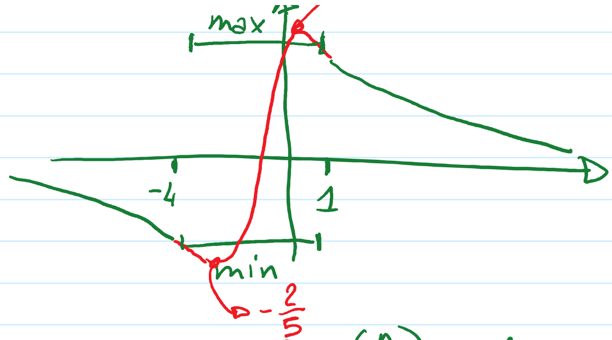
$f_{-4} = \frac{1-4}{1+16} = -\frac{3}{17}$

$f_{-3} = \frac{1-3}{1+9} = -\frac{1}{5}$

$f_{-2} = - \cdot -\frac{2}{5}$

$f_{-1} = 0$

$f_1 = 1 = f_0$



$\Longrightarrow$ $\max(A) = 1$

$\min(A) = -\frac{2}{5}$

HOMEWORKS

5) $f: \mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x) = x^2 - 2x - 3$

4) $\left\{ \frac{1}{x^2+4} : x \in \mathbb{R} \right\}$

6) $\{ x \,|\, |x| < 2 \,;\, x \in \mathbb{R} \}$

7) $\left\{ (-1)^n \frac{2n+1}{2n-1} : n \in \mathbb{N}^+ \right\}$

8) $\left\{ \frac{2n}{n^2+1} : n \in \mathbb{Z} \right\}$

9) $\left\{ \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$