$$S = \left\{ -n + \sqrt{n-2} \quad n \in \mathbb{N},\ \underline{\underline{n \geq 2}} \right\}$$

Guess $-2 = \max S$

? $-2 \geq -n + \sqrt{n-2}$ $\forall n \geq 2$?

$\Updownarrow$

$\underbrace{n-2}_{\geq 0} \geq \sqrt{n-2}$

$\Updownarrow$

$(n-2)^2 \geq n-2$

$n^2 + 4 - 4n \geq n - 2$

(*) $n^2 - 5n + 6 \geq 0$

$n_{1,2} = \dfrac{5 \pm \sqrt{25-24}}{2} = 3,\ 2$

(*) $\Longleftrightarrow$ $n \leq 2$ $n \geq 3$

We have $\begin{cases} n \leq 2 \text{ or } n \geq 3 \\ \boxed{n \geq 2} \end{cases}$

2 3 4 5

$\boxed{n \geq 2}$ or $n \geq 3$

$\max S = -2 = \sup S$

"Probably" S is not lower bounded

i.e. $\inf S = -\infty$

that is $\forall A \geq 0$

$-n + \sqrt{n-2} \leq -A$ should have at least one solution

$$\Updownarrow$$

$$\sqrt{n-2} \leq \underline{(n - A)}$$

$(n \geq A)$ $\Updownarrow$

$$n-2 \leq (n-A)^2 \iff n-2 \leq n^2 + A^2 - 2nA$$

$$(**) \quad n^2 + n(-2A-1) + (A^2+2) \geq 0$$

$$\Delta = (-2A-1)^2 - 4(A^2+2) =$$

$$\cancel{4A^2} + 1 + 4A - \cancel{4A^2} - 8 =$$

$$4A - 7$$

solut. of associated in case $\boxed{\Delta \geq 0}$

$$n_{1,2} = \frac{2A+1 \pm \sqrt{\Delta}}{2}$$

i.e. $n_1 = \frac{2A+1-\sqrt{\Delta}}{2}$, $n_2 = \frac{2A+1+\sqrt{\Delta}}{2}$

it is solved by any $n \geq n_2$

if $\Delta < 0$ any n is a solution of $(**)$

Exercise

$$S = \left\{ \frac{\sqrt{n+1}}{\sqrt{n}+1} \quad n \in \mathbb{N} \quad n \geq 1 \right\}$$

Prove that 1) $\min S = \frac{\sqrt{2}}{2}$

2) $\sup S = 1$

3) is it also $\max S = 1$?

1) i) $\frac{\sqrt{n+1}}{\sqrt{n}+1} \overset{?}{=} \boxed{\frac{\sqrt{2}}{2}}$? for ~~some~~ n.

$\Updownarrow$

$$2\sqrt{n+1} = \sqrt{2}\sqrt{n} + \sqrt{2}$$

$$4n+4 = 2n+2+4\sqrt{n}$$

$$2n+2 = 4\sqrt{n}$$

$$n+1 = 2\sqrt{n}$$

$$(\sqrt{n}-1)^2 = n+1-2\sqrt{n} = 0$$

$\Updownarrow$ $\sqrt{n} = 1$ $\quad n = 1$

yes $\frac{\sqrt{2}}{2} \in S$

ii) $\frac{\sqrt{2}}{2} \le \frac{\sqrt{n+1}}{\sqrt{n} + 1}$ $\forall n \in \mathbb{N}$, $n \ge 1$

$\Updownarrow$

$\vdots$

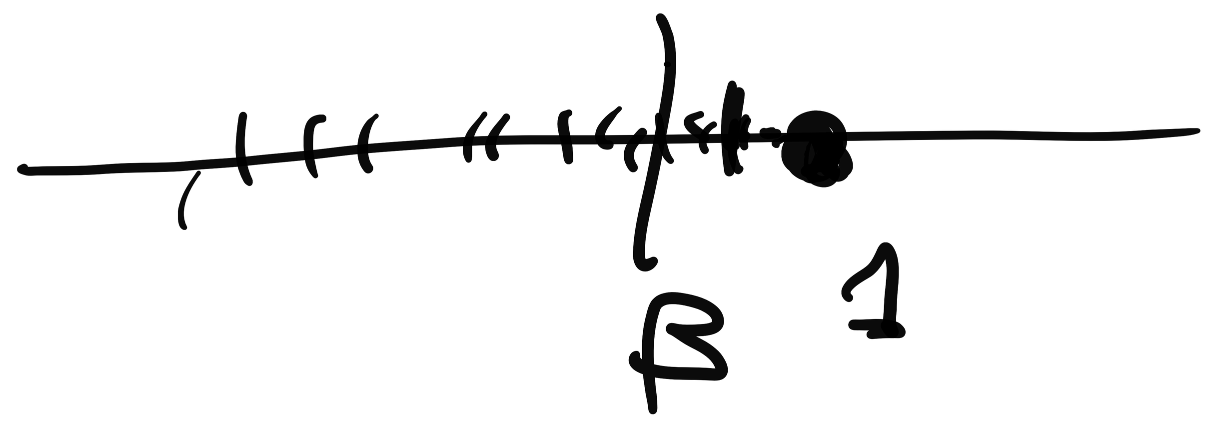
$(\sqrt{n} - 1)^2 \ge 0$ true $\forall n \in \mathbb{N}$

Let us show that $1 = \sup S$

i) $\boxed{1 \ge \frac{\sqrt{n+1}}{\sqrt{n} + 1}}$ $\forall n \in \mathbb{N}$, $n \ge 1$

ii) $\forall \beta < 1 \ \exists\, s \in S$

$\beta \le s \le 1$



$\Updownarrow$

$\exists n$ $\quad \beta \le \frac{\sqrt{n+1}}{\sqrt{n} + 1} \le 1$

? i) $\Longleftrightarrow$ $\sqrt{n} + 1 \ge \sqrt{n+1}$

$\cancel{n} + 2\sqrt{n} \cancel{+1} \ge \cancel{n+1}$

$2\sqrt{n} \ge 0$ o.k $\forall n \in \mathbb{N}$

ii) $\circledast$ $\beta \le \frac{\sqrt{n+1}}{\sqrt{n} + 1}$ for at least one n

if $\beta \le 0$ - $\star$ is trivial

if $1 > \beta > 0$

$$\star \iff \beta^2 \le \frac{n+1}{n+1+2\sqrt{n}}$$

$$\Updownarrow$$

$$\beta^2 n + \beta^2 + 2\beta^2\sqrt{n} \le n+1$$

$$\Updownarrow$$

$$n(1-\beta^2) - 2\beta^2\sqrt{n} + (1-\beta^2) \geq 0$$

$y = \sqrt{n}$

$$y^2\underbrace{(1-\beta^2)}_{>0} - 2\beta^2 y + (1-\beta^2) \geq 0$$

there $\exists\, C$ s.t.

$y \geq C$ is a solution.

<u>Choose one</u> of those y

$n = y^2$ not enough

choose a suitable y.

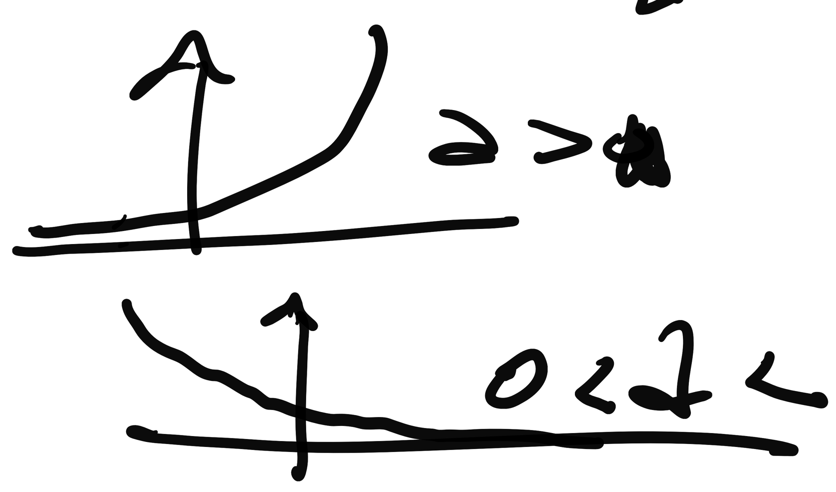
We have defined elementary function

Power to the α $\quad f: x \longmapsto x^\alpha$

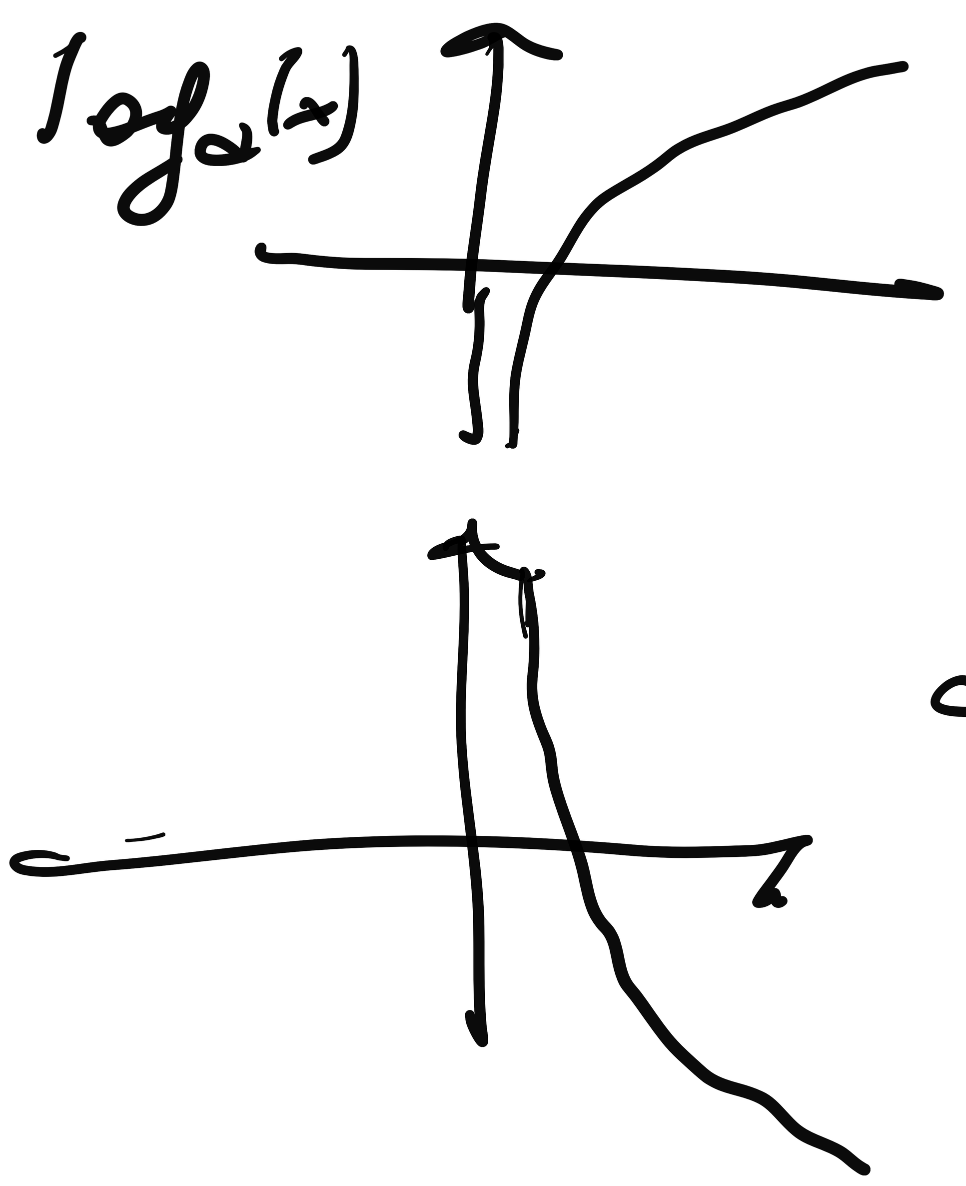
$\alpha \in \mathbb{R}$ $\quad [0,+\infty[\longrightarrow [0,+\infty[$

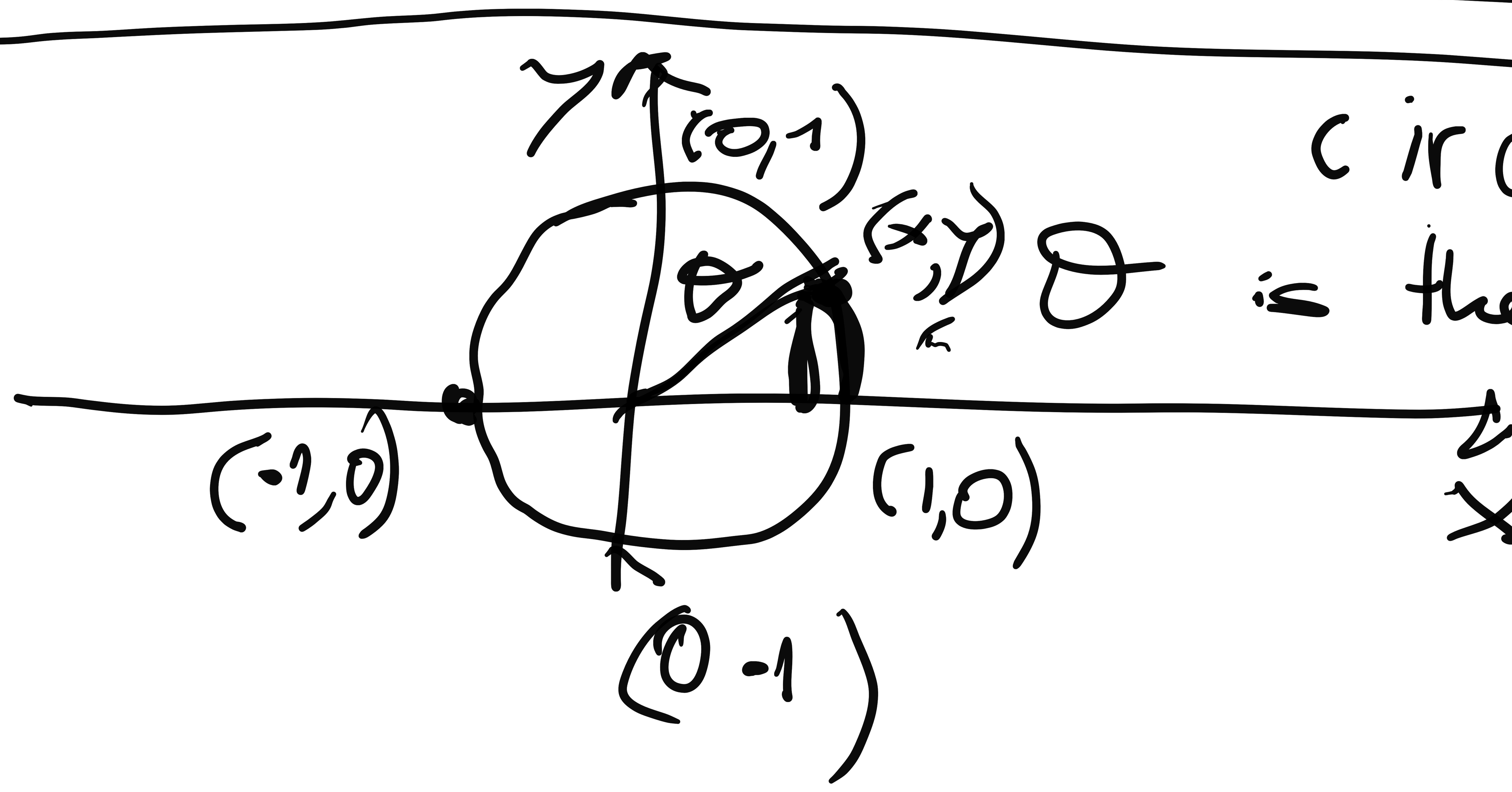
exponential with base $a>0$ $\quad x \longmapsto a^x$

graph:



logarithm is the inverse of exponential when $a \neq 1$, $a>0$

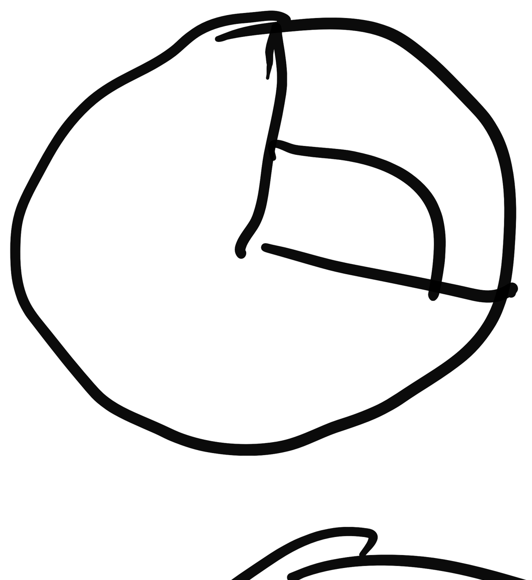




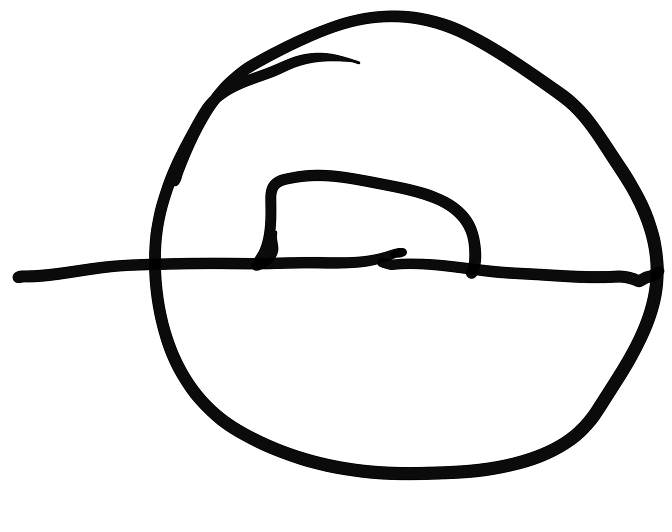
circle of radius 1

θ is the lenght of this arc.

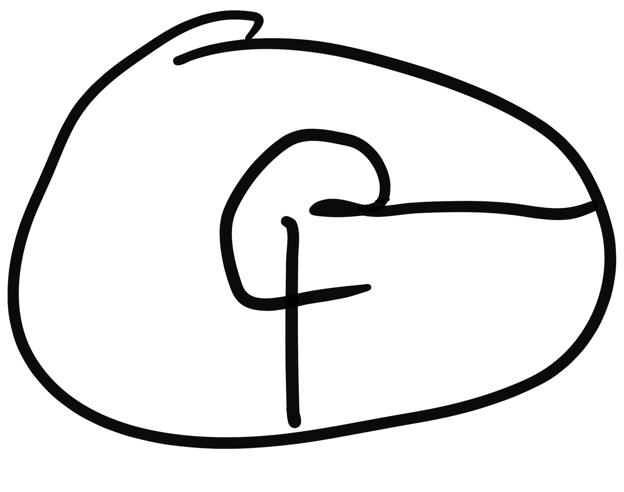
2π is the whole circle



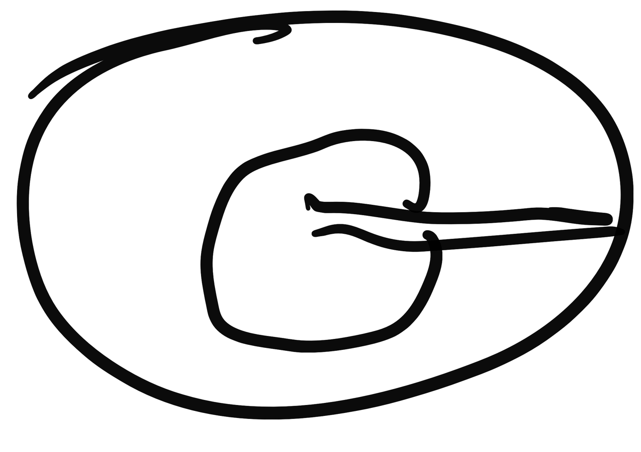
$\frac{\pi}{2}$



π



$\frac{3}{2}\pi$



2π

$\sin\theta = y$ if (x,y) are the Cartesian coordinates of the point corresponding to θ

$\cos\theta = x$

$\theta = [0, 2\pi]$

$$\sin(\theta + 2\pi) := \sin\theta$$

$$\cos(\theta + 2\pi) := \cos\theta$$

more generally define

$$\sin(\theta + 2k\pi) = \sin(\theta)$$

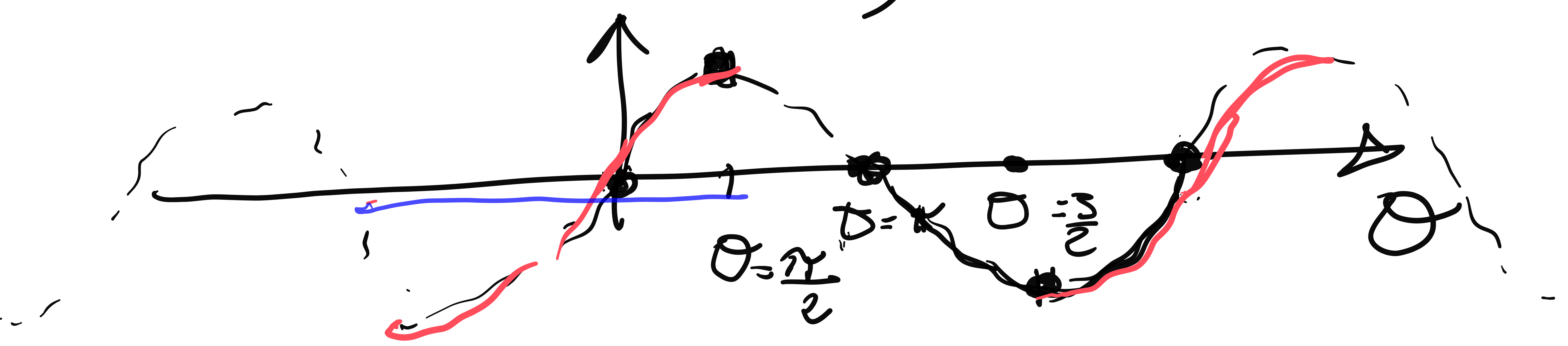
$$\forall \theta \in [0, 2\pi]$$
$$\forall k \in \mathbb{Z}$$

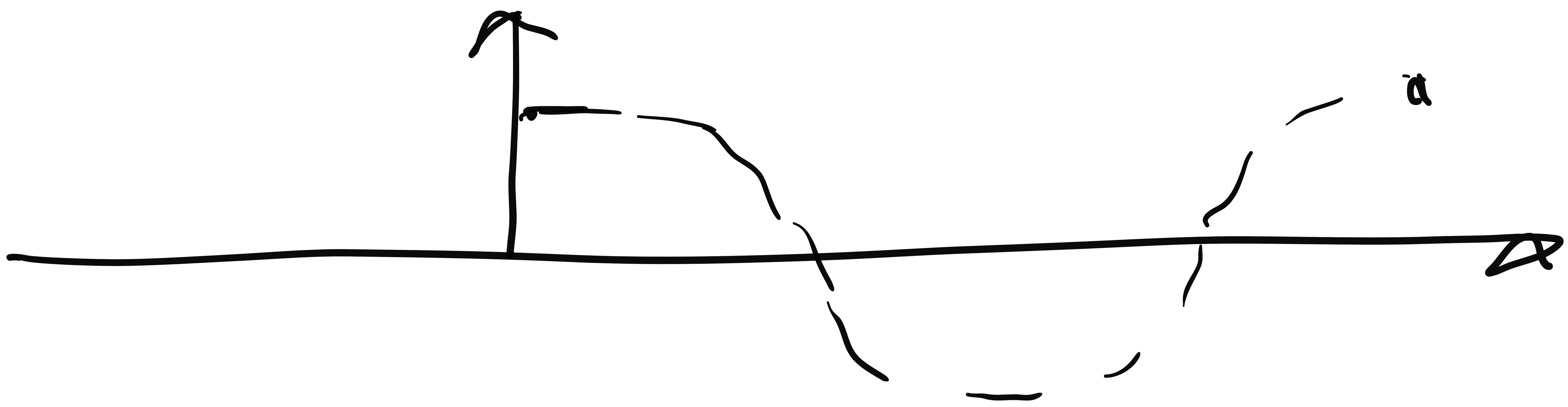
More simply one says that a function $f: \mathbb{R} \to \mathbb{R}$ is periodic with period T if

$$f(x+T) = f(x) \quad \forall x \in \mathbb{R}$$

In the ~~case~~ $f = \sin$ $T = 2\pi$

$$\sin(\theta + 2\pi) = \sin(\theta) \quad \forall \theta \in \mathbb{R}$$

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