Lesson 5 - 6/10/2022

$\boxed{1}$ Linearize $\ddot{x} = -\sin x - \dot{x}$ at equilibria $(\bar{x}, 0)$ with $\bar{x} \in [0, 2\pi)$.

First order $\rightarrow$ $\begin{cases} \dot{x} = v \\ \dot{v} = -\sin x - v \end{cases}$ $\Rightarrow$ $J(x,v) = \begin{pmatrix} 0 & 1 \\ -\cos x & -1 \end{pmatrix}$

Equilibria ? $\Leftrightarrow$ $X(\bar{x}, \bar{v}) = (0,0)$ $\Leftrightarrow$ $\begin{cases} \bar{v} = 0 \\ \sin \bar{x} = 0 \end{cases}$

$\Leftrightarrow$ $\bar{x} = 0$ and $\bar{x} = \pi$.

$\underbrace{J(0,0)}_{\text{First eq.}} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \rightarrow \begin{cases} \dot{x} = v \\ \dot{v} = -x - v \end{cases} \rightarrow \boxed{\ddot{x} = -x - \dot{x}}$

$\underbrace{J(\pi,0)}_{\text{Second eq.}} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \rightarrow \begin{cases} \dot{x} = v \\ \dot{v} = x - \pi - v \end{cases} \rightarrow \boxed{\ddot{x} = x - \pi - \dot{x}}$

The previous ex. is an example of linearization of a 2nd order diff. eq.

$\boxed{\ddot{x} = Y(x, \dot{x})}$ $\Leftrightarrow$ $\dot{v} = Y(x, v)$

At first order $\begin{cases} \dot{x} = v \\ \dot{v} = Y(x, v) \end{cases}$

In this case equilibria are always of type $(\bar{x}, 0)$ such that $\underline{Y(\bar{x}, 0) = 0}$ $\bar{x}$ is an eq. configuration

$\Rightarrow J(\bar{x}, 0) = \begin{pmatrix} 0 & 1 \\ \partial_x Y(\bar{x}, 0) & \partial_y Y(\bar{x}, 0) \end{pmatrix}$

$$\Rightarrow \boxed{\begin{cases} \dot{x} = v \\ \dot{v} = \partial_x Y(\bar{x},0)(x-\bar{x}) + \partial_y Y(\bar{x},0)\, v \end{cases} \quad \text{linearization around } (\bar{x},0) \quad (\varphi.)}$$

2nd step: Study in details linear systems.

$$\begin{cases} \dot{z} = Az \\ z(0) = z_0 \end{cases} \qquad \begin{array}{l} z \in \mathbb{R}^n \\ A \ n\times n \text{ matrix} \end{array}$$

We recall that the solution is given by the matrix exponential

[Consider the 1-dim case: $\begin{cases} \dot{x} = ax & a \in \mathbb{R} \\ x(0) = x_0 & x \in \mathbb{R} \end{cases}$

$\Rightarrow \varphi^t(x_0) = x_0 e^{at}$.

Recall that $e^a = 1 + a + \frac{a^2}{2} + \frac{a^3}{3!} + \dots \quad a \in \mathbb{R}$].

Analogously in dim n.

Def $A = n \times n$ matrix.

$$e^A := \mathbb{1} + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots = \sum_{k=0}^{+\infty} \frac{A^k}{k!} \qquad (*)$$

$\| \quad$ $\exp A$

We need to check that this def. is well-posed! In order to prove this fact, we recall that the set of $n\times n$ matrices (in $\mathbb{R}$) is a Banach space with the norm:

$$\|A\| := \sup_{\substack{x \in \mathbb{R}^n \\ x \neq \underbrace{0}_{(0,\dots,0)}}} \frac{|Ax|}{|x|} = \sup_{|x|=1} |Ax|$$

n-times

Moreover, it is a Banach algebra: $\|AB\| \leq \|A\| \cdot \|B\|$

Lemma e^A is well defined that is the series $(*)$ is convergent.

Proof $\|e^A\| = \left\| \sum_{k=0}^{+\infty} \frac{A^k}{k!} \right\| \leq \sum_{k=0}^{+\infty} \frac{\|A^k\|}{k!} \leq$

$$\underset{\substack{\downarrow \\ \text{Banach algebra}}}{\leq} \sum_{k=0}^{+\infty} \frac{\|A\|^k}{k!} = e^{\|A\|} < +\infty .$$

$\boxtimes$

Prop $\begin{cases} \dot{z} = Az \\ z(0) = z_0 \end{cases} \rightsquigarrow \varphi^t(z_0) = e^{tA} z_0$

Proof Directly

Clearly, $\varphi^0(z_0) = z_0$

$$\frac{d}{dt}\left(\varphi^t(z_0)\right) = \frac{d}{dt}\left(e^{tA} z_0\right) =$$

$$= \frac{d}{dt}\left(\sum_{k=0}^{+\infty} \frac{1}{k!}(tA)^k\right) z_0 =$$

$$= \frac{d}{dt}\left(1 + tA + \frac{(tA)^2}{2!} + \dots\right) z_0 =$$

$$= \frac{d}{dt}\left(\sum_{k=1}^{+\infty} \frac{1}{k!}(tA)^k\right) z_0 =$$

$$= \left[\sum_{k=1}^{+\infty} \frac{1}{\underbrace{\cancel{k!}}_{(k-1)!}} \cancel{k}\, t^{k-1} \underbrace{A^k}_{A A^{k-1}}\right] z_0 = \underbrace{\left[\sum_{k=0}^{+\infty} \frac{1}{k!} t^k A^k\right] A z_0}_{= \varphi^t(z_0)} =$$

$$= A \varphi^t(z_0)$$

this means that $d/dt\left(\varphi^t(z_0)\right) = A \varphi^t(z_0) \iff$

$\varphi^t(z_0)$ is the unique solution of the linear v.f. (with starting cond. z_0). $\boxtimes$

Properties / exercises on the matrix exp.

$\boxed{1}$ A, P $m \times m$ matrices. P invertible. Then

$$\exp(P^{-1}AP) = P^{-1}\exp(A)\,P$$

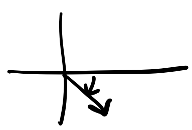
Proof $\exp(P^{-1}AP) = \sum_{k=0}^{+\infty} \frac{1}{k!} \underbrace{(P^{-1}AP)^k} = P^{-1}\exp(A)\,P.$

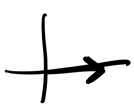
$$\underbrace{(P^{-1}A\cancel{P})(\cancel{P^{-1}}A\cancel{P})\cdot \ldots \cdot(\cancel{P^{-1}}AP)}_{k \text{ times}} = P^{-1}A^kP \qquad \boxtimes$$

$\boxed{2}$ $A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} = \beta \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{=B}$

$$\Rightarrow \quad e^{tA} = \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}$$

$\downarrow$ clockwise rotation of angle βt

$(1,0) \rightarrow$ 



$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$B^3 = -B \qquad B^4 = -B^2$

$B^5 = B$

$B \quad B^2 \quad -B \quad -B^2 \quad B \quad B^2 \ldots$

Consequently

$$e^{tA} = \sum_{k=0}^{+\infty} \frac{1}{k!}(tA)^k = \sum_{k=0}^{+\infty} \frac{1}{k!} t^k \beta^k B^k = \dots =$$

$$\underset{\downarrow}{=} \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}$$

Maclaurin series of $\sin(\beta t)$ and $\cos(\beta t)$

$\boxed{3}$ $A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \Rightarrow e^{tA} = \begin{pmatrix} \cos(\omega t) & \sin(\omega t)/\omega \\ -\omega \sin(\omega t) & \cos(\omega t) \end{pmatrix}$

A comes from $\ddot{x} = -\omega^2 x \Rightarrow \begin{cases} \dot{x} = v \\ \dot{v} = -\omega^2 x \end{cases} \Rightarrow$

$$\Rightarrow \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}_{=A} \begin{pmatrix} x \\ v \end{pmatrix}$$

We conjugate by $P = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 1/\omega \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}_{A} \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

like in Ex. 2

From ex. 1, we know that

$$\exp(tP^{-1}AP) = P^{-1}\exp(tA)P \Rightarrow$$

$$\exp(tA) = P \underbrace{\exp(tP^{-1}AP)}_{\to \text{Ex 2.}} P^{-1}$$

Therefore

$$\exp(tA) = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/\omega \end{pmatrix} =$$

$$= \begin{pmatrix} \cos(\omega t) & \sin(\omega t)/\omega \\ -\omega \sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

Example

- Solve the linear system $\dot{x} = Ax$, where $A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$
- Graph the phase-portrait as $a \in \mathbb{R}$, showing qualitative differences.

Solution

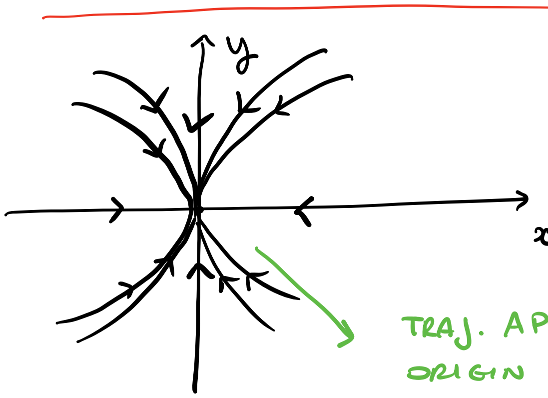
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{cases} \dot{x} = ax \\ \dot{y} = -y \end{cases}$$

$\to$ Eqs are uncoupled.

$$\begin{cases} x(t) = x_0 e^{at} \\ y(t) = y_0 e^{-t} \end{cases} \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2 \text{ initial point}$$

- $y(t)$ decays exponentially.
- When $a<0$ also $x(t)$ decays exponentially $\Rightarrow$ All trajectories approach the origin as $t \to +\infty$.

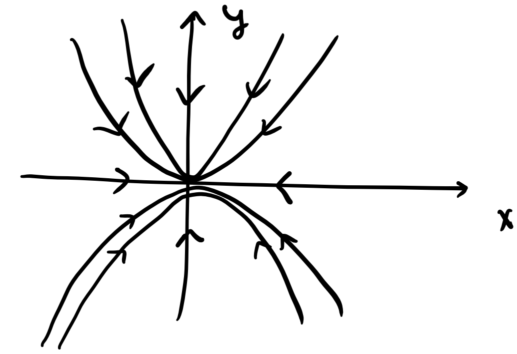
However THE DIRECTION OF APPROACH DEPENDS ON THE SIZE OF $a<0$ COMPARED TO -1!



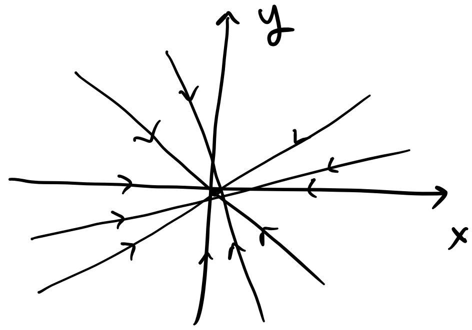
TRAJ. APPROACH THE ORIGIN TG TO THE SLOWER DIRECTION!

$a < -1$

$\downarrow$ $x(t)$ decays more rapidly than $y(t)$



$-1 < a < 0$



$a = -1$

$\downarrow$

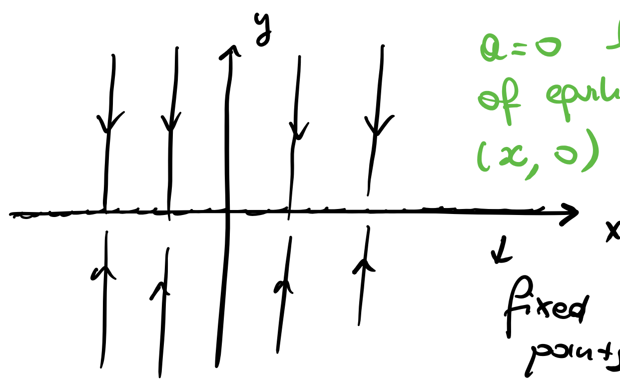
$$\begin{cases} x(t) = x_0 e^{-t} \\ y(t) = y_0 e^{-t} \end{cases}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$\boxed{a<0}$ $(0,0)$ stable (attractor)

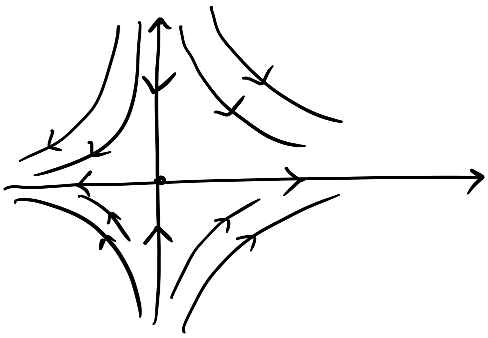
$a = 0$

$$\begin{cases} x(t) = x_0 \\ y(t) = y_0 e^{-t} \end{cases}$$



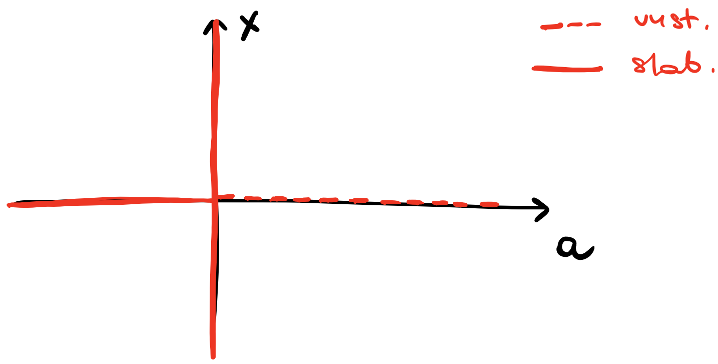
$a=0$ line of equilibria $(x,0)$ stable

$a > 0$



$a>0$ $\exists !$ eq. $(0,0)$ unstable

By consid. only the eq. configuration $x \in \mathbb{R}$, the bif. diagram is the following :



Classification of linear systems, A 2×2 diagonalizable

$$\begin{cases} \dot{z} = Az \\ z(0) = z_0 \end{cases}$$

$z \in \mathbb{R}^2$

A, 2×2 matrix diagonal.

We know: $\varphi^t(z_0) = \underbrace{e^{tA}}_{\exp(tA)} z_0$

A diag. (on $\mathbb{R}$ or $\mathbb{C}$) $\Leftrightarrow$ A has 2 eigenvectors (on $\mathbb{R}$ or $\mathbb{C}$) linearly independent.

Recall the previous example: x and y axes play a crucial role since they are <u>INVARIANT</u> lines for the dynamics.

For the general case $\dot{z} = Az$, we would like to find an analog to these axes. In particular, we seek for trajectories of this form:

$$\boxed{z(t) = e^{\lambda t} v} \qquad v \in \mathbb{C}^2, \quad \lambda \in \mathbb{C}$$

To find conditions on v and λ, we impose that $z(t)$ is a solution of $\dot{z} = Az$.

$$\lambda \cancel{e^{\lambda t}} v = A \cancel{e^{\lambda t}} v \Leftrightarrow Av = \lambda v$$

$\boxed{\text{The desired invariant lines exist iff } v \text{ is an eigenvector of } A \text{ with eigenvalue } \lambda}$

First case $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \neq 0$.

eigenvalues of A

This means that A has 2 real eigenvectors $v_1, v_2 \in \mathbb{R}^2$ linearly independent.