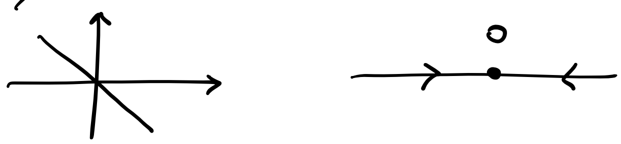
Lesson 4 - 05/10/2022

Bifurcations - Exercises -

$\boxed{1}$ $\dot{x} = \mu x$, $x, \mu \in \mathbb{R}$

$X_\mu(x) = \mu x$

$\mu < 0$

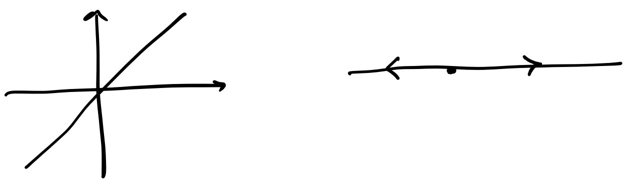


1 eq. attractor (stable).

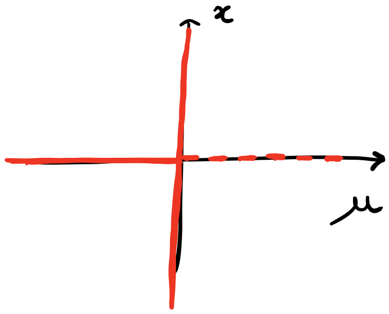
$\mu = 0$

every point on $\mathbb{R}$ is a stable eq.

$\mu > 0$



1 eq. repeller (unstable)



— = stab.

- - - - = unst.

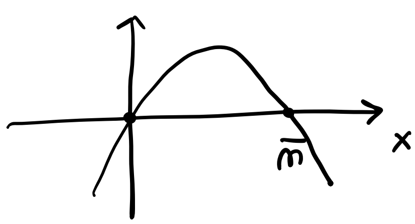
b.f. diag.

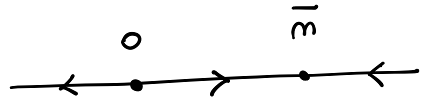
$\boxed{2}$ Logistic model with constant hauling parameter $p \geq 0$.

$$\dot{x} = K\left(1 - \frac{x}{\bar{m}}\right)x - p$$

$$X_p(x) = K\left(1 - \frac{x}{\bar{m}}\right)x - p$$

$\underline{p = 0}$

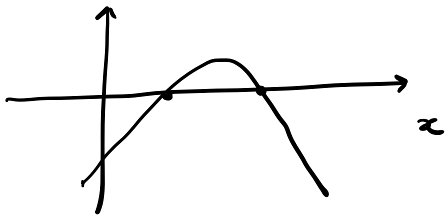




Two eq. 0 = repeller (unstable)

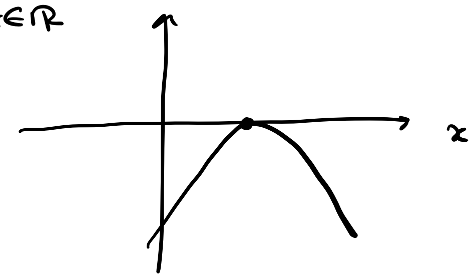
$\bar{m}$ = attract. (stable)

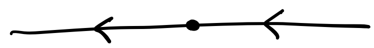
$0<p<\bar{p}$



Two eq. one attractor (on the right) and one repeller (on the left).

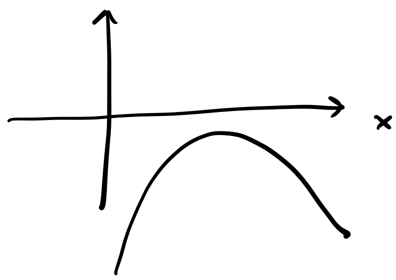
$p=\bar{p} \to \bar{p} = \max_{x\in\mathbb{R}} X_0(x)$





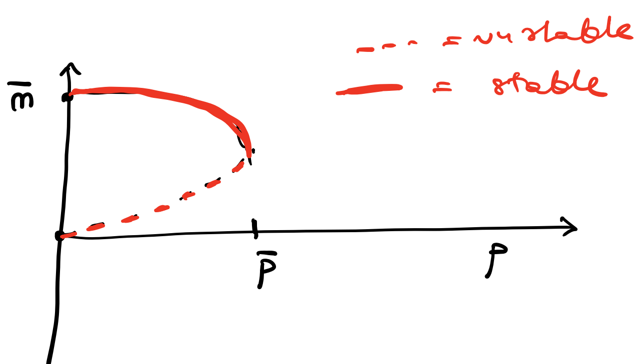
One ep. no attracting no repelling.

$p>\bar{p}$



No equilibria.

Bifurcation diagram ?!

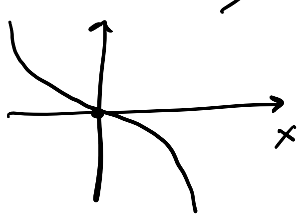


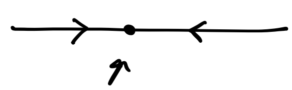
$\boxed{3}$ $\dot{x} = \underbrace{x}_{x=0}(\mu - x^2)$, $x,\mu \in \mathbb{R}$.

other equilibria dep. on the sign of μ!

$\mu - x^2 = 0 \iff x^2 = \mu$ solutions iff $\mu \geq 0$

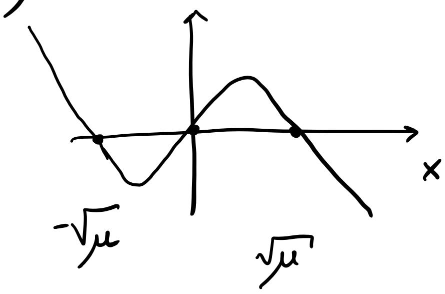
- $\mu \leq 0$: 1 ep. $(x=0)$

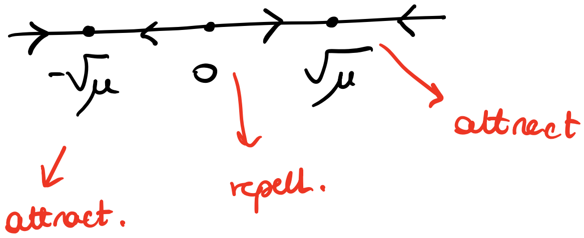


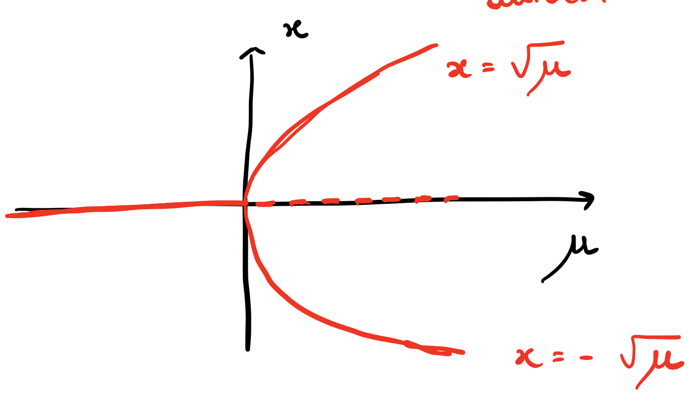


attractor (stable)

• $\mu > 0$: 3 equilibria ($x = 0$ and $\mu = x^2 \iff x = \pm\sqrt{\mu}$)



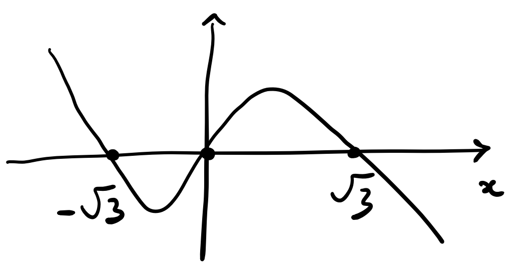


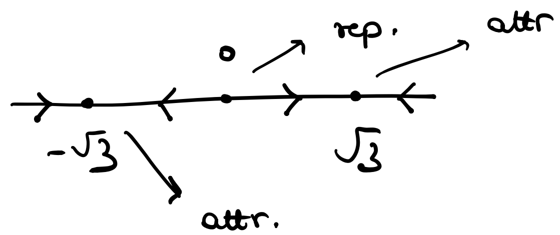


Bifurc. diagr.

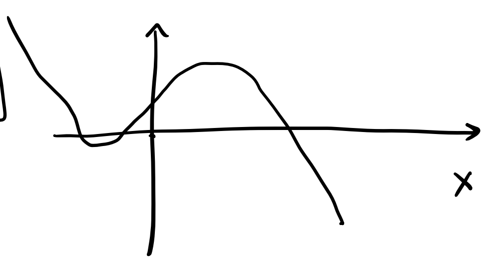
$\boxed{4}$ $\dot{x} = \mu + \underbrace{x - \frac{1}{3}x^3}_{\circ}$ $\mu \in \mathbb{R}$, $x \in \mathbb{R}$

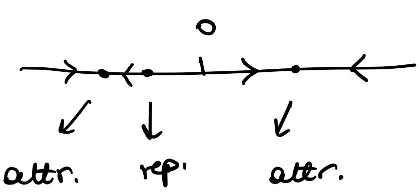
$\boxed{\mu = 0}$ $X_0(x) = x\left(1 - \frac{1}{3}x^2\right) = 0$ $\nearrow x = 0$, $\searrow x = \pm\sqrt{3}$



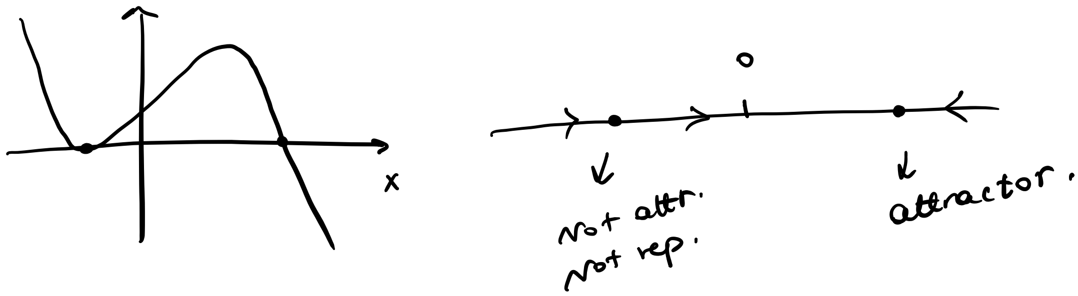


$\boxed{0 < \mu < \bar{\mu}}$

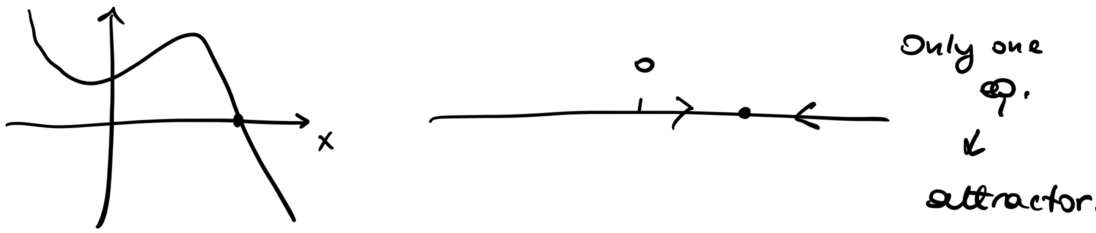




$\boxed{\mu = \bar{\mu}}$

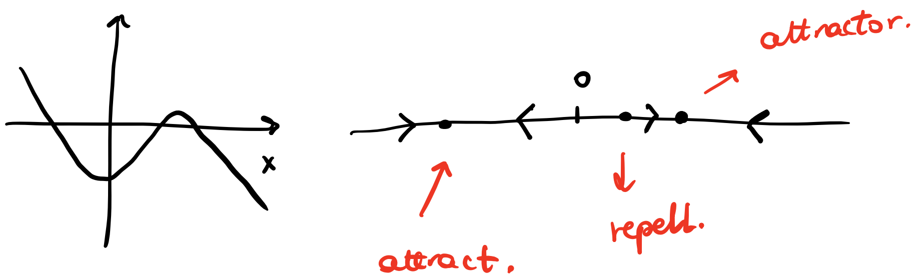


$\boxed{\mu > \bar{\mu}}$

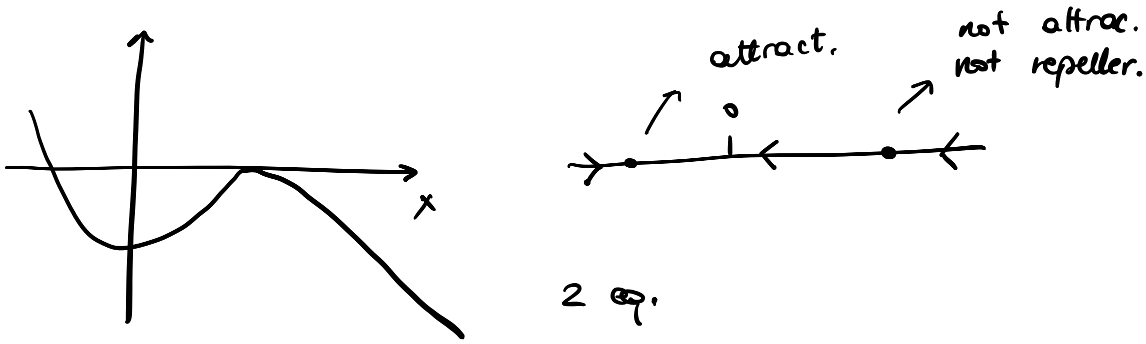


Now consider negative paramet. $\mu < 0$.

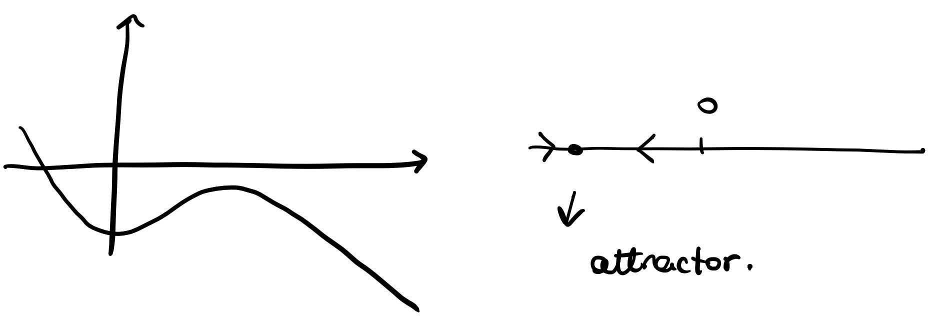
$\bar{\bar{\mu}} < \mu < 0$



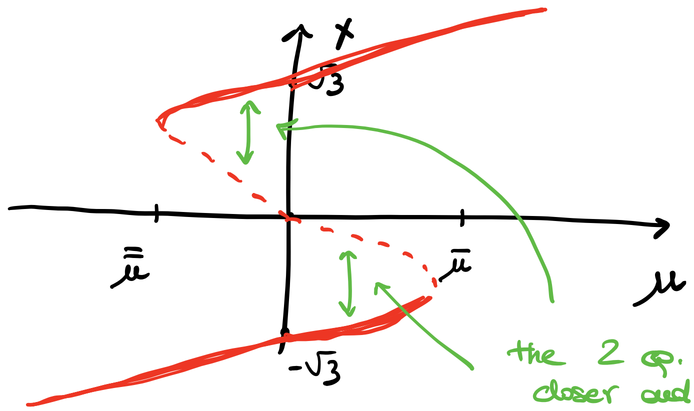
$\mu = \bar{\bar{\mu}}$



$\mu < \bar{\bar{\mu}} < 0$



Bif. diagr.



the 2 eq. become closer & closer and collapse in a unique one at the critical parameter $\mu = \bar{\mu}$ (or $\mu = \bar{\bar{\mu}}$)

$\boxed{5}$

$$\dot{x} = x - 2\mu x^2 + x^3 = x(1 - 2\mu x + x^2)$$

for $\underline{\mu > 0}$, $x \in \mathbb{R}$.

$$X_\mu(x) = x(1 - 2\mu x + x^2)$$

$x = 0$ eq.

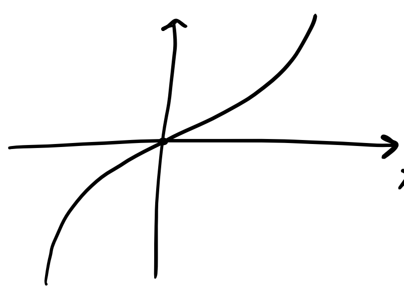
$x^2 - 2\mu x + 1 = 0$ has solutions iff $\Delta = 4\mu^2 - 4 > 0$

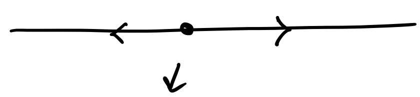
$\iff \mu < -1$ or $\boxed{\mu > 1}$

If $\mu > 1$ then $x_{1,2} = \dfrac{2\mu \pm 2\sqrt{\mu^2 - 1}}{2} = \mu \pm \sqrt{\mu^2 - 1}$

both positive! $(\mu > 1)$

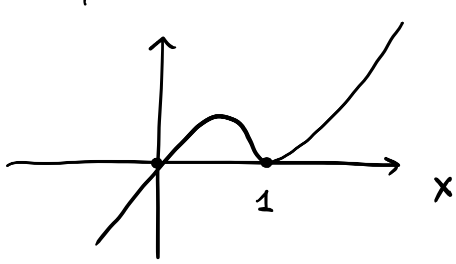
$\boxed{0 < \mu < 1}$

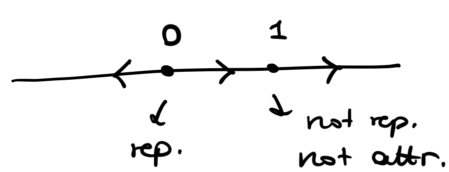




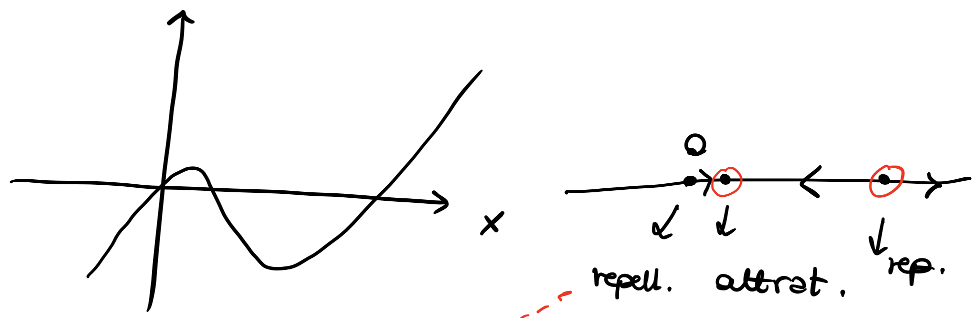
One eq, repeller

$\boxed{\mu = 1}$

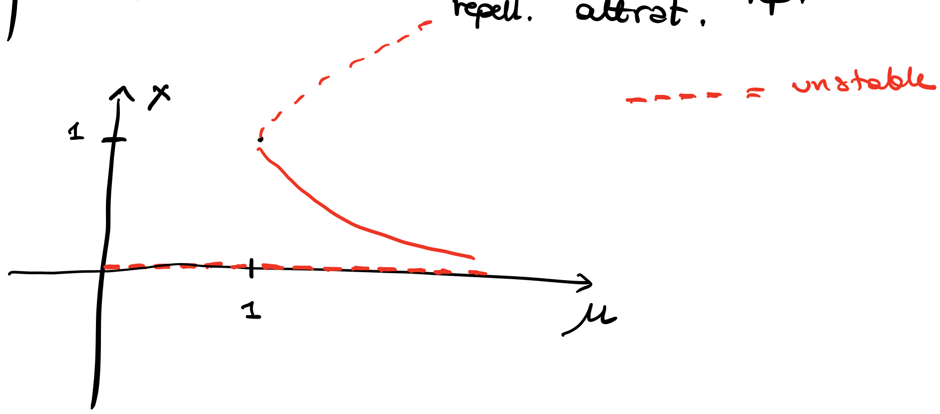




$\boxed{\mu > 1}$



Bif. diag.



—x—x—x—

$\dot{x} = X(x)$, $x \in \mathbb{R}$.

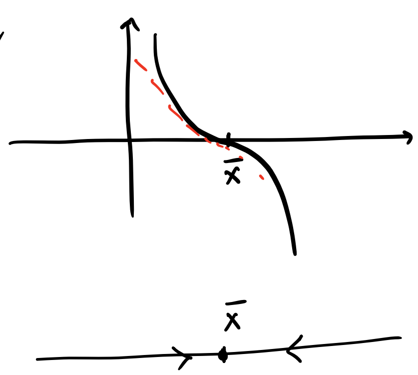
Role of $X'(\bar{x})$ for the quality of the eq $\bar{x}$?!

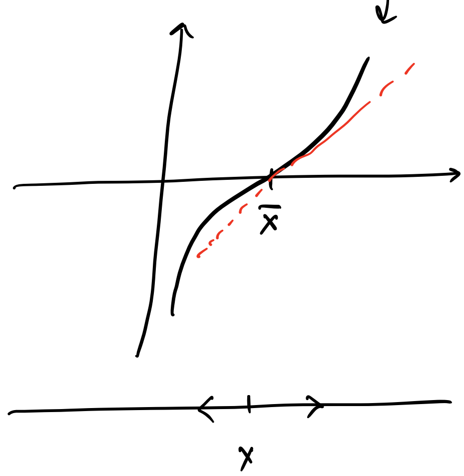
Prop $\dot{x} = X(x)$, $x \in \mathbb{R}$.

$\bar{x} \in \mathbb{R}$ equilibrium, that is $X(\bar{x}) = 0$. Then:

if $X'(\bar{x}) < 0$ then $\bar{x}$ is an attractor.

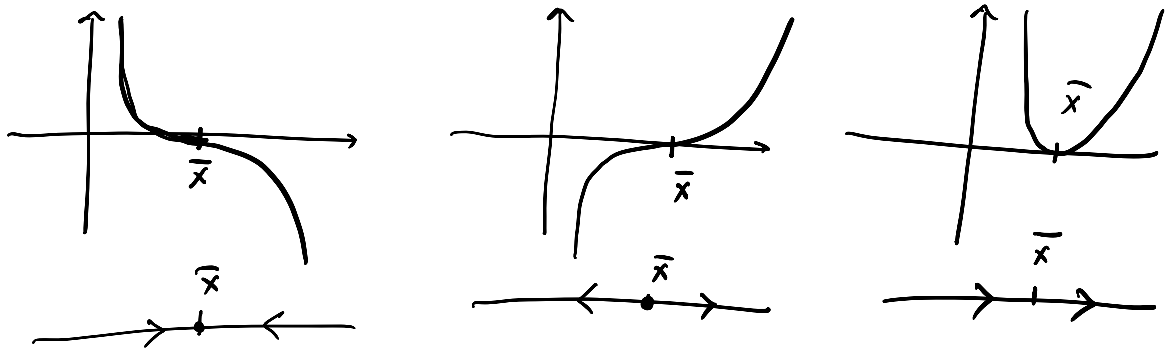
if $X'(\bar{x}) > 0$ then $\bar{x}$ is a repeller.





If $x'(\bar{x}) = 0$ then we cannot conclude nothing about the polity of $\bar{x}$. If fact all these three diff. cases can occur.

$X'(\bar{x}) = 0$



Conclusion for 1-dim v.f. on $\mathbb{R}$: The sign of $x'(\bar{x})$ ($\bar{x} \in \mathbb{R}$ equilibrium) gives us informations about the "quality" of the eq. itself.

Aim of next lectures : Can we extend this conclusion also for $n > 1$ v.f. ?

We need to linearize a v.f. around an eq. $\bar{x} \in \mathbb{R}^n$, ($n > 1$)

$\dot{x} = X(x)$, $x \in \mathbb{R}^n$, $X \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$

$\bar{x} \in \mathbb{R}^n$ ep. that is $X(\bar{x}) = 0$. Consider the Taylor expansion around $\bar{x} \in \mathbb{R}^n$:

$$X(x) = \underbrace{X(\bar{x})}_{=0 \ (\bar{x} \text{ is ep.})} + \boxed{A}(x - \bar{x}) + o((x-\bar{x})^2)$$

where A with $A_{ij} = \dfrac{\partial X_i}{\partial x_j}(\bar{x})$ → Jacobian matrix of X evaluated in $\bar{x}$.

We want to obtain possible informations on $\dot{x} = X(x)$ in a neighb. of an equilibrium $\bar{x} \in \mathbb{R}^n$ by studying the linearized v.f. $\dot{x} = A(x - \bar{x})$

$\Longleftrightarrow \quad \dot{z} = Az$

$z = x - \bar{x}$

Steps

Learn linearing a v.f. 1)

study linear v.f. (local) 2)

state general results about relations between $\dot{x} = X(x)$ and $\dot{x} = A(x-\bar{x})$ around $\bar{x} \in \mathbb{R}$ 3)

① 1-dim $\quad \dot{x} = kx - gx^2 = x(k - gx) \qquad X'(x) = k - 2gx$

$\bar{x} = 0, \quad \dot{x} = kx$

$\bar{x} = \frac{k}{g}, \quad \dot{x} = -k\left(x - \frac{k}{g}\right)$

② 2-dim

$$\begin{cases} \dot{x} = -x + xy \\ \dot{y} = -x - y + y^2 \end{cases} \qquad (x, y) \in \mathbb{R}^2$$

Equilibria ?

$$\begin{cases} -x + xy = x(y-1) = 0 \to x = 0 \text{ or } y = 1 \\ -x - y + y^2 = 0 \end{cases}$$

$x = 0, \; y = 0$ or $y = 1$ $\qquad (0,0)$ and $(0,1)$

Only 2 equilibria.

Linearization around $(0,0)$.

$$J(x,y) = \begin{pmatrix} -1+y & x \\ -1 & -1+2y \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{cases} \dot{x} = -x \\ \dot{y} = -x - y \end{cases}$$

Linearization around $(0,1)$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y-1 \end{pmatrix} \rightarrow \begin{cases} \dot{x} = 0 \\ \dot{y} = -x+y-1 \end{cases}$$

3

$$\begin{cases} \dot{x} = yz \\ \dot{y} = 2xz \\ \dot{z} = -3xy \end{cases}$$

3-dim

Equilibria : $(x,0,0)$, $(0,y,0)$, $(0,0,z)$

$\forall\, x,y,z \in \mathbb{R}$.

For ex, the linearization around $(0,0,1)$ is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z-1 \end{pmatrix}$$

— x —

Linearize $\ddot{x} = \underbrace{-\sin x}_{} \underbrace{-\dot{x}}_{}$ at equilibria with $\bar{x} \in [0, 2\pi)$
conf.

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\sin x - v \end{cases} \rightarrow J(x,v) = \begin{pmatrix} 0 & 1 \\ -\cos x & -1 \end{pmatrix}$$