This is a writing pen

Theorem: The sum of the first n numbers is $\frac{n(n+1)}{2}$, i.e.,
$$\left.\begin{array}{l} 1 + 2 + \dots + n = \frac{n(n+1)}{2} \\ \qquad \forall n \in \mathbb{N} \end{array}\right\} \mathcal{P}_n$$

Proof (by induction)

- is $\mathcal{P}_1$ true? $1 = \frac{1 \cdot (1+1)}{2}$
- assume $\mathcal{P}_n$ we want "$\Longrightarrow \mathcal{P}_{n+1}$"

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

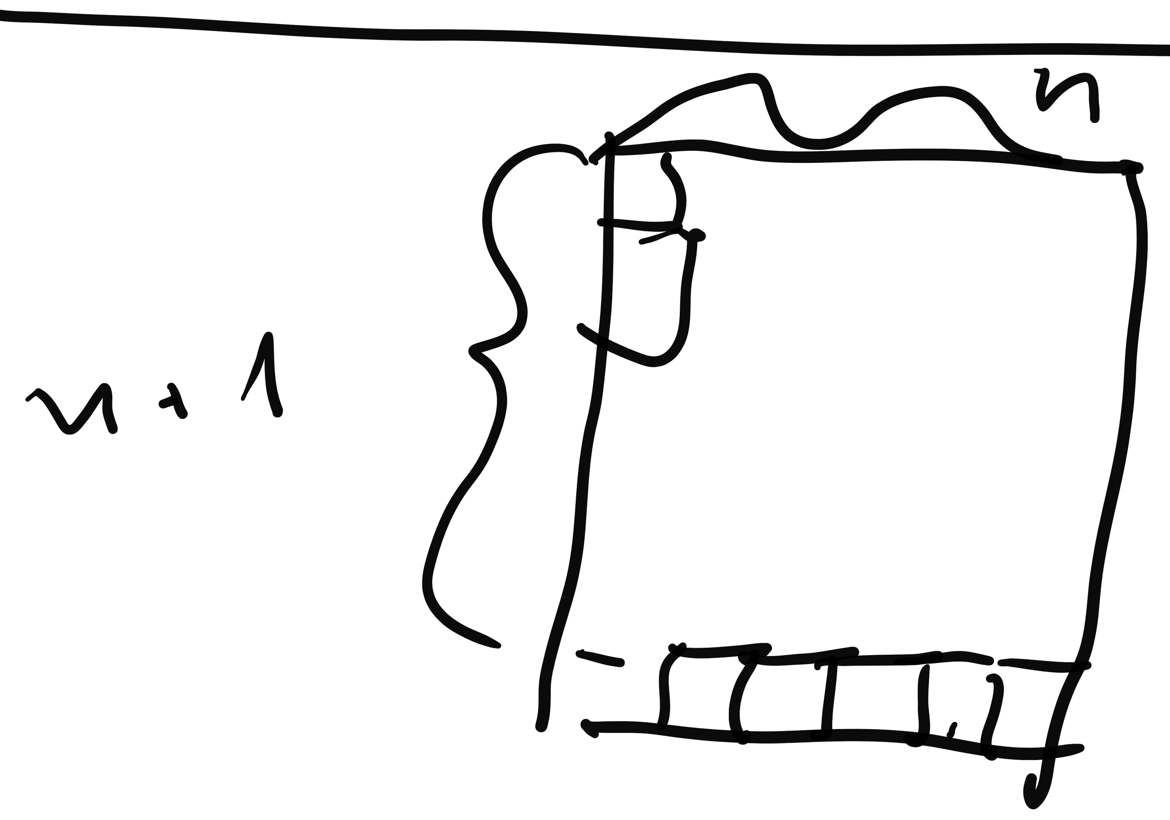
we want to prove $\mathcal{P}_{n+1}$, i.e.,

$$1 + 2 + \dots + n + (n+1) = \frac{(n+1)(n+2)}{2}$$

$$\underbrace{1+2+\dots+n}_{\frac{n(n+1)}{2}} + (n+1) = \frac{n(n+1)}{2} + (n+1) =$$

$$\frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

q.e.d



Real numbers

Call interval $\underline{I} \subset \mathbb{R}$

if .

$\forall \; x_1 < x_2 \quad x_1 \in I \;\; x_2 \in I$

$\forall \; x$ st. $x_1 \le x \le x_2 \implies x \in I$

$f: \mathbb{R}^+_{\geq} \longrightarrow \mathbb{R} \qquad n \in \mathbb{N}$

$\quad\; \| $

$[0, +\infty[$

$f(x) = x^n$ power of exponent n.

Is it injective?

$$x_1 \neq x_2 \overset{?}{\Longrightarrow} f(x_1) \neq f(x_2)$$

$$\boxed{\begin{array}{c} 0 \le x_1 < x_2 \\ \Downarrow \\ (*) \quad x_1^n < x_2^n \end{array}}$$

$$x_1^n \neq x_2^n$$

is it true? Let us prove (*) by induction:

It is true for $n=1$.

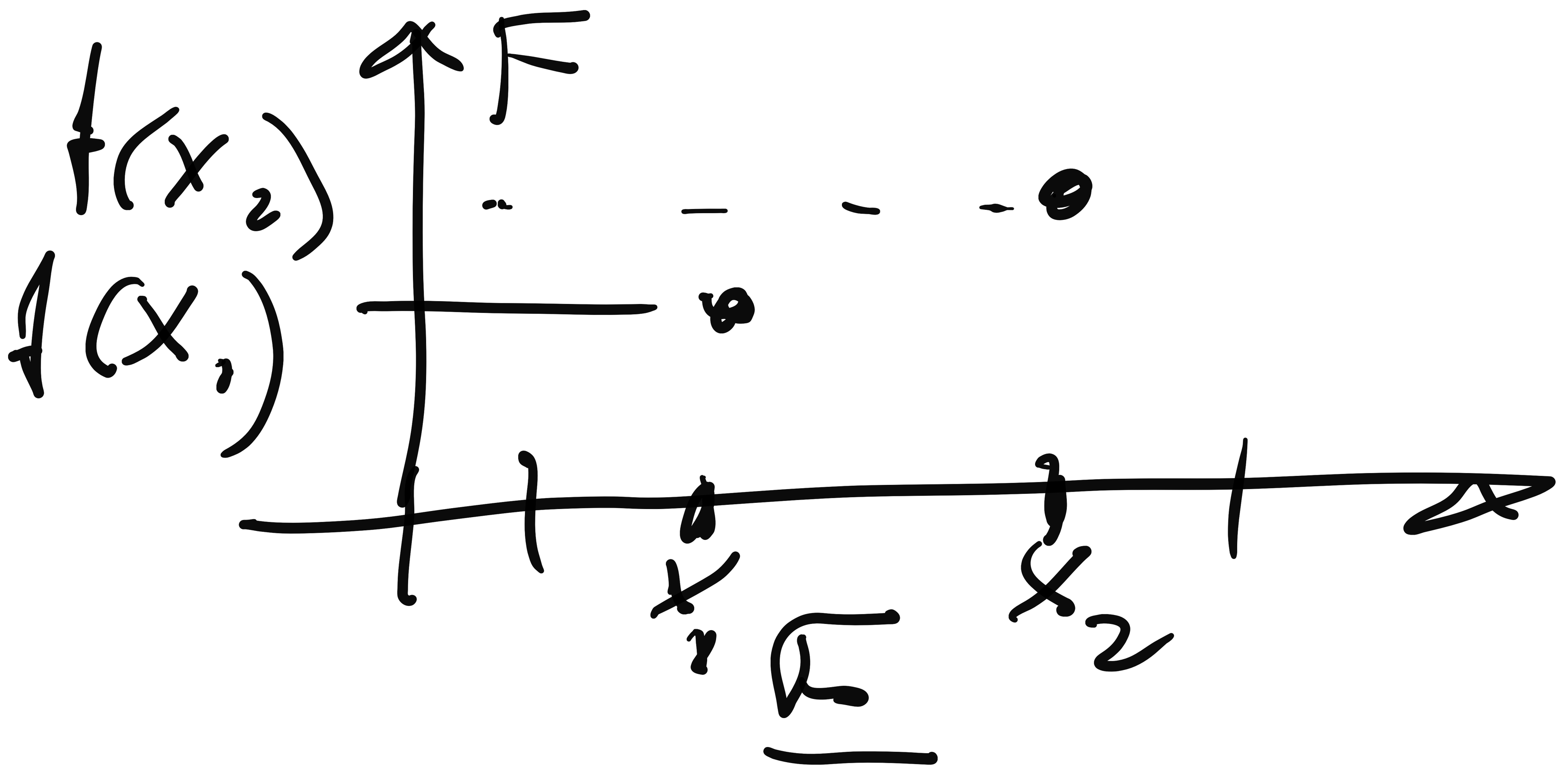
- Suppose $x_1^n < x_2^n \overset{?}{\Longrightarrow} x_1^{n+1} < x_2^{n+1}$

$$x_1^{n+1} = x_1 x_1^n < x_2 x_2^n = x_2^{n+1}$$

Definition: $g : E \longrightarrow F$, $E \subset \mathbb{R}$, $F \subset \mathbb{R}$

g is (strictly) increasing if

$$\forall\, x_1, x_2 \in E \quad x_1 < x_2 \Longrightarrow f(x_1) \le f(x_2) \quad (<)$$



Hence $f(x) = x^n$ $\left(x \in [0, +\infty)\right)$ is strictly increasing.

We have used

Theorem: If f is strictly increasing then it is injective

Proof if $x_1 \neq x_2$, assume $x_1 < x_2$

$\xRightarrow{f \text{ is strictly increasing}}$ $f(x_1) < f(x_2)$, $\Rightarrow f(x_1) \neq f(x_2)$

q.e.d.

Def $f: E \to F$ is decreasing (strictly) if $\forall\ x_1, x_2 \in E$

$x_1 < x_2$

$\Longrightarrow f(x_1) \geq f(x_2)$

$(>)$

Theorem Strictly decreasing

$\Downarrow$

injective

Proof is similar to the previous one

$f(x) = x^2$ $(x \geq 0)$ is strictly increasing so it is injective.

$f: \mathbb{R}^+_{\geq} \to \mathbb{R}^+_{\geq}$

Is it surjective?

$y = 0$ $x = 0$ $0 = x^2 = y = 0$

$y = 1$ $x = 0$ $1 = x^2 = y = 1$

$y = \frac{1}{9}$ $x = \frac{1}{3}$ $x^2 = \frac{1}{9}$

$y = 25 \qquad x = 5$

$y = 2 \qquad x = \sqrt{2} := \sup \{ q \in \mathbb{Q} \; q^2 < 2 \}$

$y = 7 \qquad x = \sqrt{7} := \sup \left\{ \begin{array}{l} q \in \mathbb{Q} \\ q^2 < 7 \end{array} \right\}$

$(\sqrt{7})^2 = 7$

$y > 0 \qquad x = \sqrt{y} := \sup \left\{ \begin{array}{l} q \in \mathbb{Q} \\ q^2 < y \end{array} \right\}$

$y = 16 \qquad x = 4$

We have proved that

$x \longmapsto x^2$ is surjective

$\mathbb{R}^+_{\geq} \longrightarrow \mathbb{R}^+_{\geq}$

but also

$x \longmapsto x^n$ " " (for every fixed n)

Hence, for every $n \in \mathbb{N}$, $n \geq 1$,

$f(x) = x^n$ is bijective

$f: \mathbb{R}^+_{\geq 0} \longrightarrow \mathbb{R}^+_{\geq 0}$.

Definition: $f: A \to B$

$g: B \to A$ is called the inverse of f if $\forall x \in A \quad g(f(x)) = x$

$\forall y \in B \quad f(g(y)) = y$

Question: Has every function f an inverse? No.

Example: $f: \mathbb{R} \to \mathbb{R}^+_{\geq 0}$

$f(x) = x^2 \qquad \nexists g$ s.t. $g(x^2) = x$

$g: \mathbb{R}^+ \to \mathbb{R} \qquad g(y) = \sqrt{y}$

$\tilde{g}(y) = -\sqrt{y}$

this f has NOT an inverse

Example $\hat{f}\ \mathbb{R}^+_{\geq 0} \to \mathbb{R}^+_{\geq 0}$

$\hat{g}(y) = \sqrt{y} \qquad \hat{g}(\hat{f}(x)) = x$

$\hat{f}(\hat{g}(y)) = y$

better

Definition $f: A \to B$

bijective $\Longrightarrow$ there exists an inverse g

$g: B \to A$

$g(y) := x$ s.t. $f(x) = y$

By surjectivity this x exists

" injectivity " " is unique

Definition V, W two sets

$$\underbrace{V \times W}_{\text{PRODUCT of } V \text{ and } W} := \{(v,w) \quad v \in V \quad w \in W\}$$

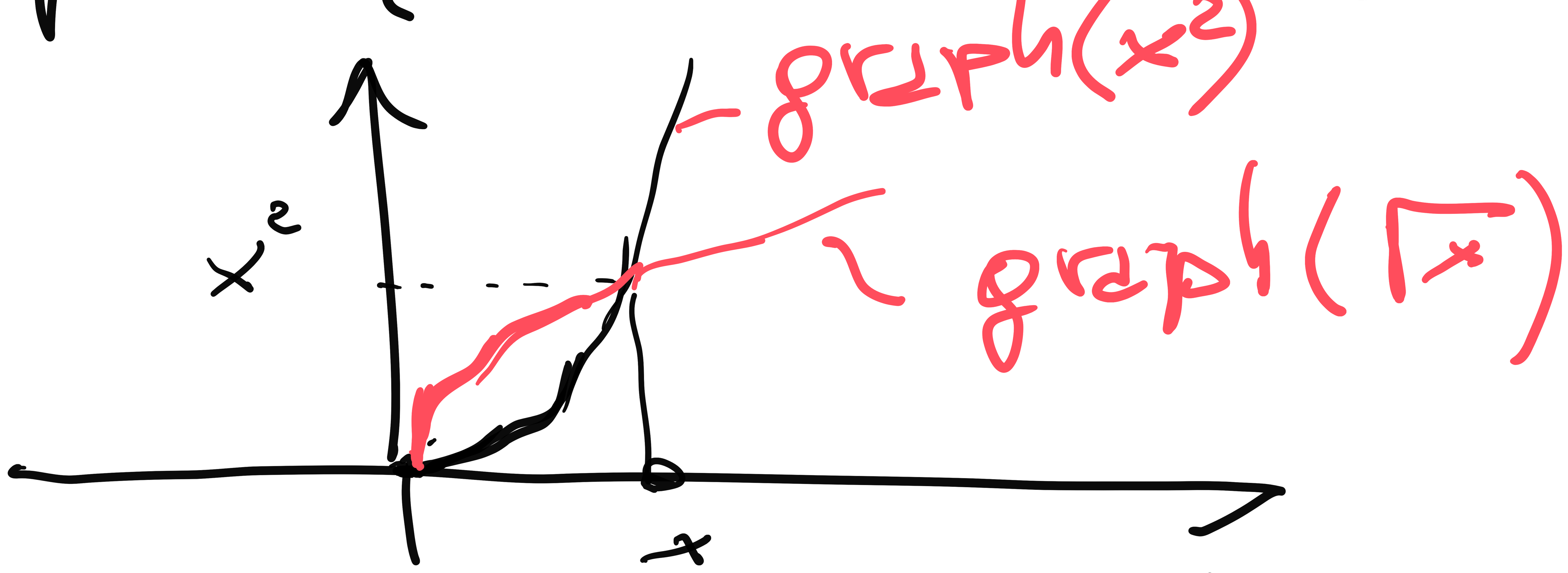
Definition: $f : A \to B$

the graph of f is defined as

$$\text{graph}(f) = \{(a,b) \in A \times B : f(a) = b\}$$

Example: $f(x) = x^2$ $\quad f : \mathbb{R}^+_{\geq} \to \mathbb{R}^+_{\geq 0}$

$$\text{graph} = \{(x, x^2), \ x \in \mathbb{R}^+_{\geq}\}$$

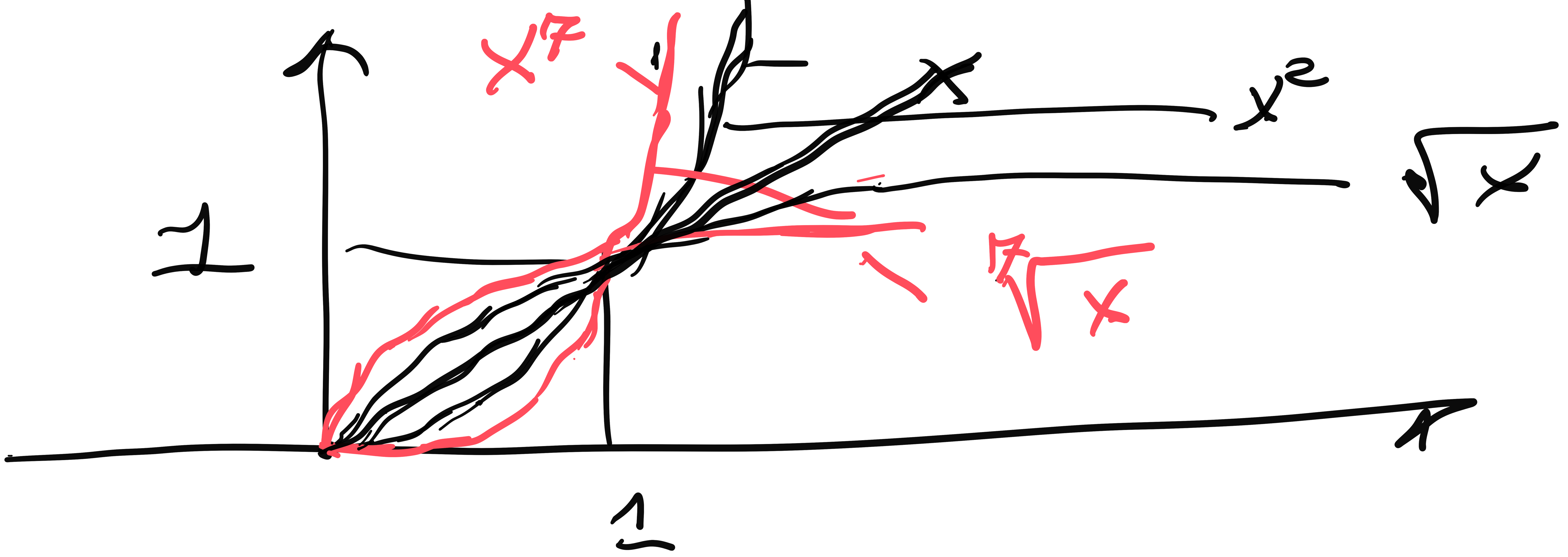


How we represent the inverse of a funtion.

$$\to \text{graph}(f^{-1}) = \{(y, f^{-1}(y)) \quad y \in B\}$$

$$= \{(\underline{f(x)}, x) \quad x \in A\}$$

$$\text{graph}(f) = \{(x, \underline{f(x)}) \quad x \in A\}$$



$10 = a > 0$

$x \mapsto a^x$

$a \in \mathbb{R}$

$a > 0$

$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots a}_{n \text{ times}}$

$a^{\frac{1}{n}} := \sqrt[n]{a}$

m, n have no common factors

$\boxed{n > 0}$

$$a^{\frac{m}{n}} := \left(a^{\frac{1}{n}}\right)^m \quad \text{if } m > 0$$

if $m < 0$ $\quad \left(a^{\frac{m}{n}}\right) = \frac{1}{a^{-\frac{m}{n}}}$

We have defined a^q for every $q \in \mathbb{Q}$

If $r \in \mathbb{R}$

$$a^r = \sup\{a^q \;\; q \in \mathbb{Q}\}_{q \le r}$$

$a \geq 1$

$0 \le a \le 1$ $\qquad \inf \{ a^q \; q \in \mathbb{Q} \}_{q \le r}$

$a \ge 0$

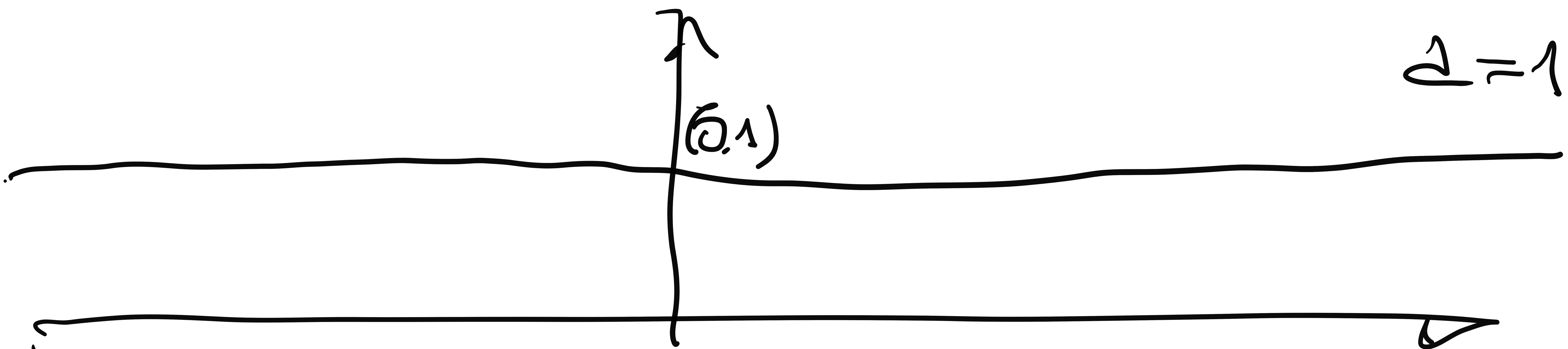
$$\begin{array}{ccc} x & \longmapsto & a^x \\ \cap & & \\ \mathbb{R} & \longmapsto & \boxed{\mathbb{R}^*_{>} =]0,+\infty[} \end{array}$$

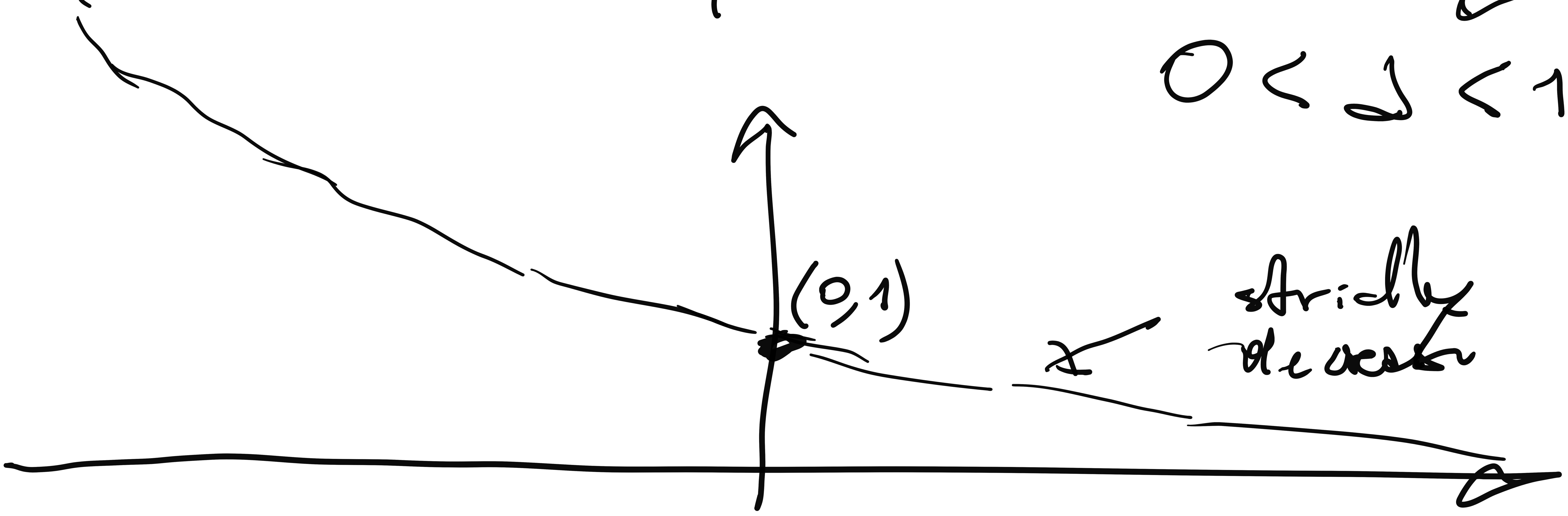
"Exponential map"

If $a > 1$ $\quad x \mapsto a^x = \exp_a(x)$

is strictly increasing

is surjective





$a > 0 \quad a \neq 1$ the inverse of $x \longmapsto a^x$ is called the "logarithm with base a"