MATH4U - 29/09/2022

Monday, September 26, 2022 6:22 PM

CONTACT INFO : DAVIDE TOMASELLA
davide.tomasella@studenti.unipd.it

EXERCISE

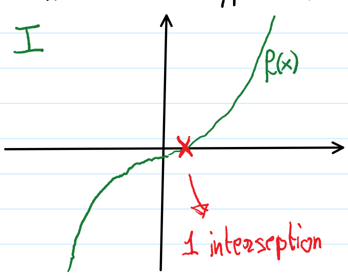
Given the equation $x^3 - x^2 - 8x + \lambda = 0$, find λ such that there are only two distinct solutions.

HINT : Consider the function $y = f(x) = x^3 - x^2 - 8x + \lambda$, it is a cubic. What are the intersection of the plot with the x-axis ? In which cases we have only two interseptions ?

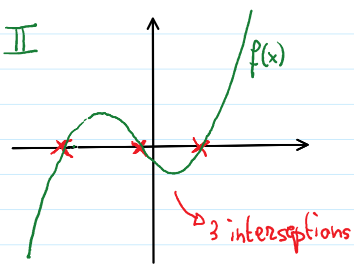
SOLUTION

We know two types of cubic functions :

I

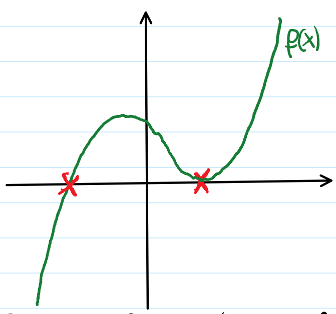


II



The interseptions are the solutions of our problem $f(x) = 0$

How can we obtain two solutions ?



A type-II function with an opportune offset has only two **DISTINCT** solutions

$\Downarrow$

In our problem, the offset is the unknown $\boldsymbol{\lambda}$.

What are the equations of type-I and type-II cubic functions?

TYPE-I : 1 interception $\Rightarrow$ 1 real solution

$f(x) = (ax^2 + bx + c)(x - d)$, $\Delta = b^2 - 4ac < 0$ | $f(x) = a(x-b)^3$

TYPE-II : 3 interceptions $\Rightarrow$ 3 real solutions

$f(x) = a(x-b)(x-c)(x-d)$ with $a \neq 0$

$\Rightarrow$ We have only **2** solutions if $b = c \neq d$

$$
\begin{aligned}
f(x) &= a(x-b)^2(x-d) \\
&\overset{!}{=} a(x^2 - 2bx + b^2)(x-d) \\
&\overset{!}{=} a(x^3 - 2bx^2 + b^2x - dx^2 + 2bdx - db^2) \\
&\overset{!}{=} a[x^3 + (-2b-d)x^2 + (b^2 + 2bd)x - db^2]
\end{aligned}
$$

$$\Updownarrow$$

$$f(x) = x^3 - x^2 - 8x + \lambda$$

$$\begin{cases} a = 1 \\ -2b - d = -1 \\ b^2 + 2bd = -8 \\ -db^2 = \lambda \end{cases} \iff \begin{cases} a = 1 \\ d = 1 - 2b \\ b^2 + 2b(1-2b) + 8 = 0 \\ \lambda = -db^2 \end{cases}$$

$$\left[\begin{array}{l} b^2 - 4b^2 + 2b + 8 = 0 \\ -3b^2 + 2b + 8 = 0 \\ b_{12} = \dfrac{-1 \pm \sqrt{1+24}}{-3} = \dfrac{1 \mp 5}{3} = -\dfrac{4}{3}, 2 \end{array}\right]$$

$$\iff \begin{cases} a = 1 \\ b = 2 \\ d = 1 - 4 = -3 \\ \lambda = 3 \cdot 4 = 12 \end{cases} \vee \begin{cases} a = 1 \\ b = -4/3 \\ d = 11/3 \\ \lambda = -\dfrac{176}{27} \end{cases}$$

$$\boxed{\text{SOLUTION :} \quad \lambda \in \left\{ 12, -\frac{176}{27} \right\}}$$

EXTRA : 1) Why do we have two possible values for λ ? (Recall the plots)

2) Do you know another method to solve the problem ?

Take-home message :

To pass this class (and, in general, any engineering program), it is **NOT** enough learn by heart the solutions of the exercises. You must understand the **THEORY** and been able to apply it in a problem never seen before.

REVIEW EXERCISES

1) $\sqrt{3(x^2-1)+10x} < 5-x$

2) $\begin{cases} \dfrac{x+2}{x} + 3x > \dfrac{5x+6}{2} \\ \dfrac{2x}{x^2-1} \leq \dfrac{x}{x-1} \end{cases}$

3) $\begin{cases} |x+3| + |y+1| = 3 \\ x + |2y-1| = 0 \end{cases}$

4) $3^{1+x} + \left(\dfrac{1}{3}\right)^{-x} \leq \sqrt{3}$

5) $2\log_2(1-x) - \log_2|x| \geq 1$

6) $\sin x(\sqrt{3}\sin x + \cos x) = 0$

7) $\cos x + 2\sin x + 2 = 0$

8) $\log(2\cos x + \sin x) < 0$

9) $\dfrac{\sin x + \sqrt{3}\cos x + 1}{\operatorname{tg} x - 1} \geq 0$

10) $\cos 2x + \cos^2\left(\dfrac{x}{2}\right) \leq -\dfrac{1}{2}$

EX1 : $\sqrt{3(x^2-1)+10x} < 5-x$

DOMAIN: $3(x^2-1)+10x \geq 0 \iff 3x^2+10x-3 \geq 0$

$x_{12} = -5 \pm \sqrt{5+9} = -5 \pm \sqrt{14}$

$3 = \sqrt{9} < \sqrt{14} < 4 = \sqrt{16}$

$x_1 \approx -8.8, \quad x_2 \approx -1.8$

$\boxed{x \leq -5-\sqrt{14} \quad \vee \quad x \geq -5+\sqrt{14}}$

SOLUTION: we consider 2 cases : $5-x \geq 0$ and $5-x < 0$

CASE I: $5-x \geq 0 \to$ we can have solutions because $\sqrt{x} \geq 0 \quad \forall x \in \mathbb{R}$

$\sqrt{3(x^2-1)+10x} < 5-x \iff \begin{cases} 5-x \geq 0 \\ 3x^2+10x-3 \geq 0 \\ 3(x^2-1)+10x < (5-x)^2 \end{cases}$

$3x^2-3+10x < x^2-10x+25$

$2x^2+20x-28 < 0$

$x^2+10x-14 < 0$

$x_{12} = -5 \pm \sqrt{25+14} = -5 \pm \sqrt{39}$

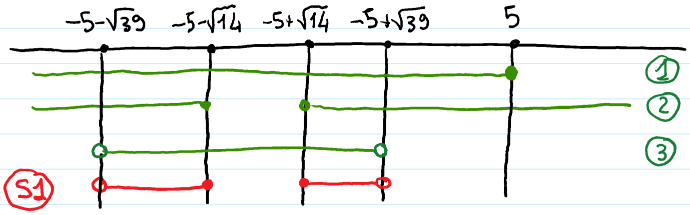
$-5-\sqrt{39} < x < -5+\sqrt{39}$

$6 = \sqrt{36} < \sqrt{39} < 7 = \sqrt{49}$

$\Rightarrow \begin{cases} -5-\sqrt{39} \approx -11.... \\ -5+\sqrt{39} \approx 1.... \end{cases}$

$\iff \begin{cases} x \leq 5 & (1) \\ x \leq -5-\sqrt{14} \ \vee \ x \geq -5+\sqrt{14} & (2) \\ -5-\sqrt{39} < x < -5+\sqrt{39} & (3) \end{cases}$

$$\Rightarrow \begin{cases} x \le 5 & ① \\ x \le -5-\sqrt{14} \ \vee \ x \ge -5+\sqrt{14} & ② \\ -5-\sqrt{39} < x < -5+\sqrt{39} & ③ \end{cases}$$



$$\boxed{S1: \ -5-\sqrt{39} < x < -5-\sqrt{14} \ \vee \ -5+\sqrt{14} < x < -5+\sqrt{39}}$$

CASE 2: $5-x<0 \longrightarrow$ we cannot have solutions

$$\underbrace{\sqrt{3(x^2-1)}}_{\ge 0} < \underbrace{5-x}_{<0} \qquad \boxed{S2: \varnothing}$$

$$\Rightarrow S = S1 \cup S2$$

$$\boxed{\text{SOLUTION}: \ -5-\sqrt{39} < x < -5-\sqrt{14} \ \vee \ -5+\sqrt{14} < x < -5+\sqrt{39}}$$

EX2 :
$$\begin{cases} \dfrac{x+2}{x} + 3x > \dfrac{5x+6}{2} & ① \\ \dfrac{2x}{x^2-1} \le \dfrac{x}{x-1} & ② \end{cases}$$

SOLUTION :

① $\dfrac{x+2}{x} + 3x > \dfrac{5x+6}{2}$ DOMAIN : $x \neq 0$

$$\frac{3x^2+x+2}{x} - \frac{5x+6}{2} > 0$$

$$\frac{6x^2+2x+4-5x^2-6x}{2x} > 0$$

$$\frac{x^2-4x+4}{2x} > 0 \Rightarrow \frac{(x-2)^2 \rightarrow N}{2x \rightarrow D} > 0$$

		0		2	
N :	+		+	0	+
D :	−	0	+		+
N/D :	−	∄	+	0	+

$$\boxed{S1: \ 0<x<2 \ \vee \ x>2}$$

② $\dfrac{2x}{x^2-1} \leq \dfrac{x}{x-1}$

$\dfrac{2x}{(x-1)(x+1)} - \dfrac{x}{x-1} \leq 0$ DOMAIN: $x \neq \pm 1$

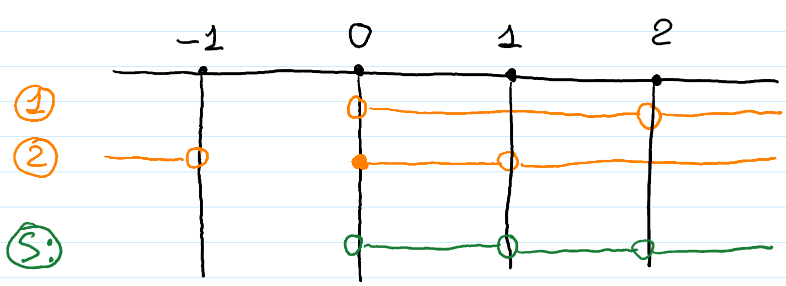
$\dfrac{2x - x^2 - x}{(x-1)(x+1)} \leq 0 \iff \dfrac{x^2-x}{(x-1)(x+1)} \geq 0$

$\dfrac{x\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} \geq 0$ ⟶ NOTE: we can semplify iff we remember the domain at the end

		-1		0	
N:	$-$		$-$	0	$+$
D:	$-$	0	$+$		$+$
$\frac{N}{D}$:	$+$	$\nexists$	$-$	0	$+$

$$\boxed{\begin{gathered} S2: \ x<-1 \ \vee \ x \geq 0 \ \wedge \ x \neq 1 \\ \Updownarrow \\ x<-1 \ \vee \ 0 \leq x < 1 \ \vee \ x>1 \end{gathered}}$$

$\Rightarrow S = S1 \cap S2$



$$\boxed{\text{SOLUTION}: \ x>0 \ \wedge \ x \neq 1,2}$$

EX3: $\begin{cases} |x+3| + |y+1| = 3 \\ x + |2y-1| = 0 \end{cases}$

SOLUTION :

① $|x+3|+|y+1|=3$ DOMAIN : $(x,y)\in\mathbb{R}^2$

CASE A: $x+3\geq 0$, $y+1\geq 0$

$|x+3|+|y+1|=3 \iff x+3+y+1=3$

$x+y+1=0 \longrightarrow y=-x-1$

CASE B: $x+3\geq 0$, $y+1<0$

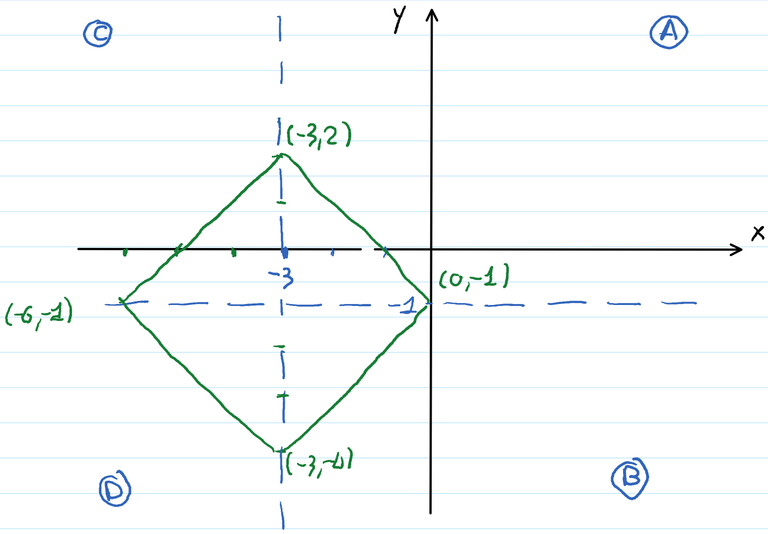
$x+3-y-1=3 \longrightarrow y=x-1$

CASE C: $x+3<0$, $y+1\geq 0$

$-x-3+y+1=3 \longrightarrow y=x+5$

CASE D: $x+3<0$ $y+1<0$

$-x-3-y-1=3 \longrightarrow y=-x-7$



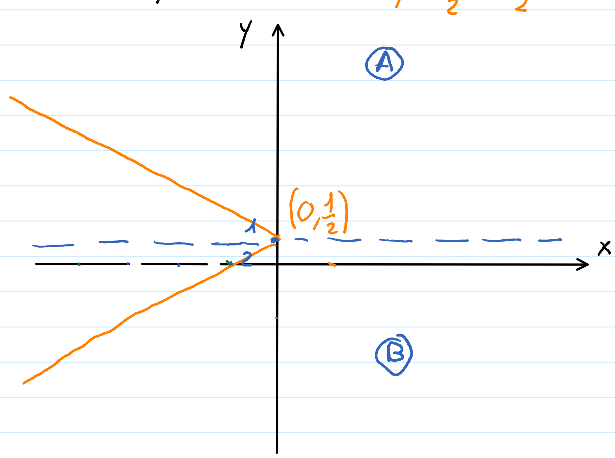
② $x+|2y-1|=0$ DOMAIN: $(x,y)\in\mathbb{R}^2$

CASE A: $2y-1\geq 0$

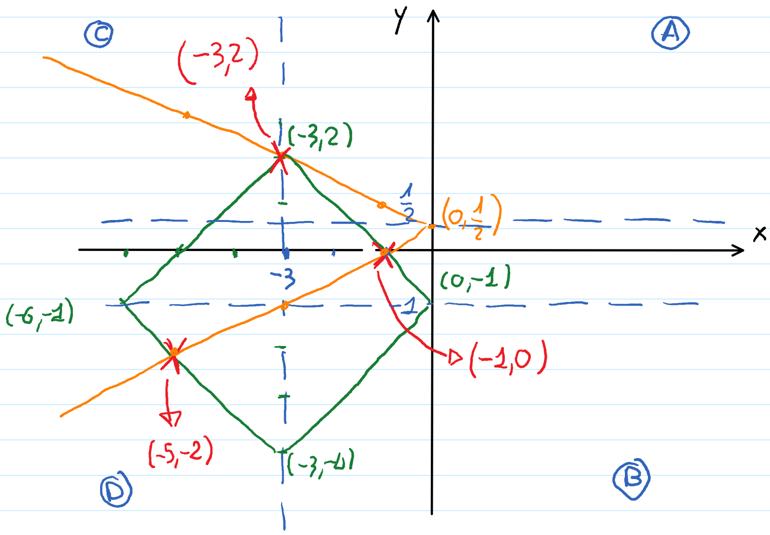
$x+2y-1=0 \longrightarrow y=-\frac{1}{2}x+\frac{1}{2}$

CASE B: $2y-1<0$

$x-2y+1=0 \longrightarrow y=\frac{1}{2}x+\frac{1}{2}$



$S = S1 \cap S2$



$$\boxed{\text{SOLUTION : } (x,y) \in \{(-5,-2);(-1,0);(-3,2)\}}$$

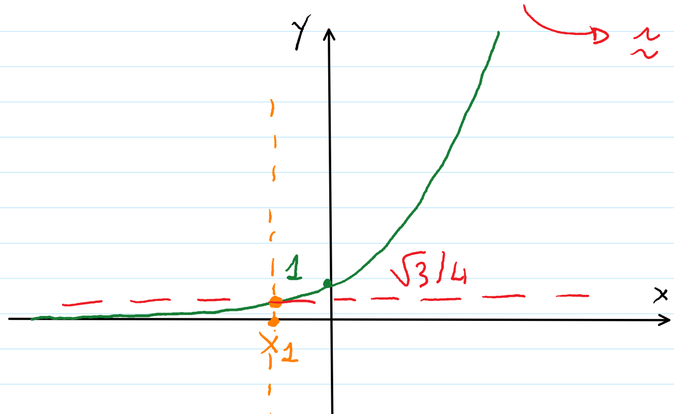
EX4: $3^{1+x} + \left(\frac{1}{3}\right)^{-x} \leq \sqrt{3}$

SOLUTION :

$$3^{1+x} + \left(\frac{1}{3}\right)^{-x} \leq \sqrt{3} \iff 3 \cdot 3^{x} + 3^{x} \leq 3^{1/2}$$

$$4 \cdot 3^{x} \leq 3^{1/2} \iff 3^{x} \leq \sqrt{3}/4$$

$$\approx \frac{1.7}{4} \approx 0.4$$



$$x_1 = \log_3 \sqrt{3}/4 = \frac{1}{2} - \log_3 4$$

$\log_3 4 \to 1.....$

$= -0.5....$

$$\boxed{\text{SOLUTION : } x \leq \frac{1}{2} - \log_3 4}$$

EXTRA: How can we solve $3^{1+x}+\left(\frac{1}{3}\right)^{x}\leq\sqrt{3}$?

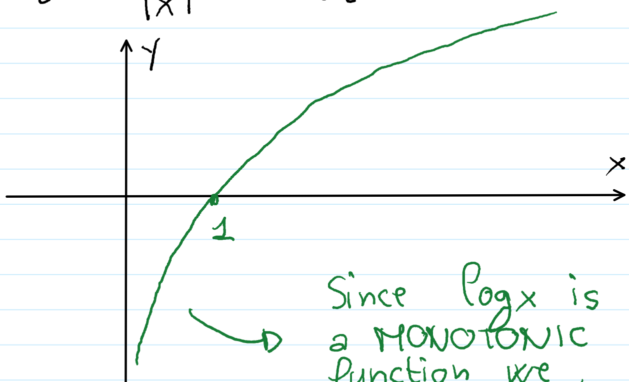
EX5: $2\log_2(1-x)-\log_2|x|\geq 1$

DOMAIN: $\begin{cases}1-x>0\\|x|>0\end{cases}\iff x<1\wedge x\neq 0$

SOLUTION:

$2\log_2(1-x)-\log_2|x|\geq 1$

$\log_2\dfrac{(1-x)^2}{|x|}\geq\log_2 2$



Since $\log x$ is a MONOTONIC function we can evaluate its argument directly

$\dfrac{(1-x)^2}{|x|}\geq 2$

CASE A: $x\geq 0$

$\dfrac{(1-x)^2}{x}-2\geq 0\iff\dfrac{x^2-2x+1-2x}{x}\geq 0$

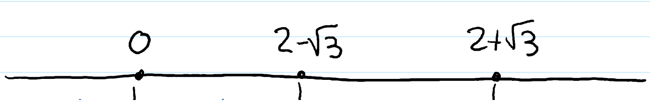
$\dfrac{x^2-4x+1}{x}\geq 0$

$x^2-4x+1=0$

$x_{1-2}=2\pm\sqrt{4-1}=2\pm\sqrt{3}$

$\sqrt{3}\approx 1.7$

$x_1\approx 0.3$, $x_2\approx 3.7$



x		0		$2-\sqrt{3}$		$2+\sqrt{3}$	
N:	$+$		$+$	0	$-$	0	$+$
D:	$-$	0	$+$		$+$		$+$
$\frac{N}{D}$:	$-$	$\nexists$	$+$	0	$-$	0	$+$

S1 : $0 < x \leq 2-\sqrt{3} \quad \vee \quad x \geq 2+\sqrt{3}$

CASE B : $x<0$

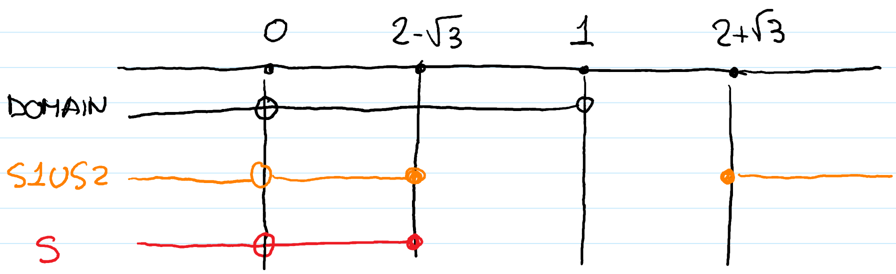
$$\frac{(1-x)^2}{-x} - 2 \geq 0 \iff \frac{x^2 \cancel{-2x} + 1 \cancel{+2x}}{x} \leq 0$$

$$\frac{x^2+1}{x} \leq 0 \quad \Delta < 0 \iff x^2+1 \geq 0 \quad \forall x$$

x		0	
N:	$+$		$+$
D:	$-$	0	$+$
$\frac{N}{D}$:	$-$	$\nexists$	$+$

S2 : $x<0$

$\implies S = (S1 \cup S2) \cap$ DOMAIN



$$\boxed{\text{SOLUTION} \quad x \leq 2-\sqrt{3} \ \wedge \ x \neq 0}$$

EX 6 : $\sin x \left(\sqrt{3}\sin x + \cos x\right) = 0$

SOLUTION :

$\sin x = 0 \quad \vee \quad \left(\sqrt{3}\sin x + \cos x\right) = 0$

SOLUTION :

$$\sin x = 0 \quad \vee \quad (\sqrt{3}\sin x + \cos x) = 0$$

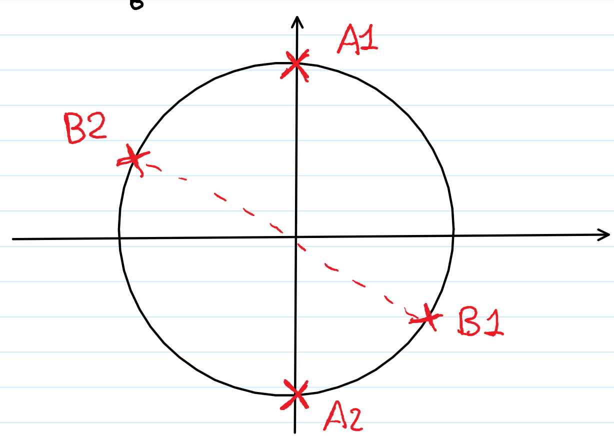
(A) (B)

(A) $\sin x = 0 \qquad x = k\pi \;, \; k \in \mathbb{Z}$

(B) $\sqrt{3}\sin x + \cos x = 0$

$$\sqrt{3}\sin x = -\cos x \iff tg\,x = -\frac{\sqrt{3}}{3}$$

$$x = -\frac{\pi}{6} + k\pi, \quad k \in \mathbb{Z}$$



$$\boxed{\text{SOLUTION :} \quad x \in \left\{-\frac{\pi}{6} + k\pi, k \in \mathbb{Z}\right\} \cup \left\{k\pi, k \in \mathbb{Z}\right\}}$$

ADVICE : Review the most common goniometric angles.

EX7 : $\cos x + 2\sin x + 2 = 0$

SOLUTION : We have a linear equation in sin and cos
$\Rightarrow$ we have to use the know substitution :

$$\cos x = \frac{1-t^2}{1+t^2} \;, \quad \sin x = \frac{2t}{1+t^2} \;, \quad tg\,x = \frac{2t}{1-t^2}$$

where $t = tg\, x/2$

$$\cos x + 2\sin x + 2 = 0 \iff \frac{1-t^2}{1+t^2} + \frac{4t}{1+t^2} + 2 = 0$$

$$\frac{1-t^2+4t+2+2t^2}{1+t^2} = 0 \iff \frac{t^2+4t+3}{t^2+1} = 0$$

$\hookrightarrow t^2+1 \neq 0 \quad \forall t$

$t^2+4t+3 = 0$

$(t+3)(t+1)=0$

$\Rightarrow$ • $t=-1 \Rightarrow tg\, x/2 = -1 \Leftrightarrow \frac{x}{2} = -\frac{\pi}{4}+k\pi,\ k\in\mathbb{Z}$

$x = -\frac{\pi}{2}+2k\pi,\ k\in\mathbb{Z}$

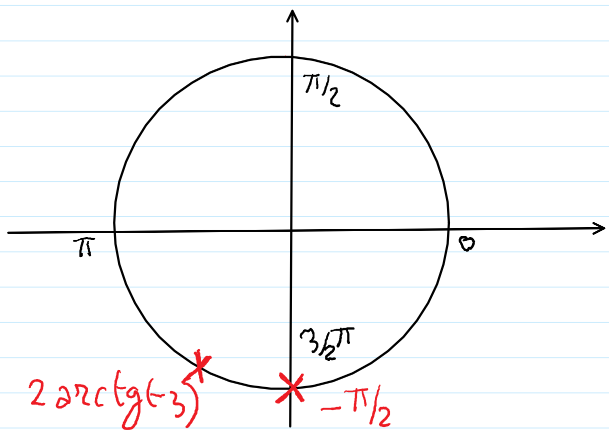
• $t=-3 \Rightarrow tg\, x/2 = -3 \Leftrightarrow \frac{x}{2} = arctg(-3)+k\pi,\ k\in\mathbb{Z}$

$x = 2\, arctg(-3)+2k\pi,\ k\in\mathbb{Z}$

$\hookrightarrow arctg(-1) = -\frac{\pi}{4} > arctg(-3) > -\frac{\pi}{2}$

$\rightarrow 2\,arctg(-3) \in \left]-\pi, -\frac{\pi}{2}\right[$

$$\boxed{\text{SOLUTION: } x = -\frac{\pi}{2}+2k\pi \ \vee\ 2\,arctg(-3)+2k\pi}$$



EXTRA: Solve the exercise with the geometric method: $X=\cos x$ and $Y=\sin x$, ...

EX8: $\log(2\cos x+\sin x)<0$

SOLUTION: Do you recall EX5?

$\log x < 0 \Leftrightarrow 0<x<1$

① $2\cos x+\sin x > 0$

$\sin x > -2\cos x$

(A) if $\cos x > 0 \Leftrightarrow -\frac{\pi}{2}+2k\pi < x < \frac{\pi}{2}+2k\pi,\ k\in\mathbb{Z}$

$\frac{\sin x}{\cos x} > -2 \Leftrightarrow tg\,x > -2$

(B) if $\cos x<0 \Leftrightarrow \frac{\pi}{2}+2k\pi < x < \frac{3}{2}\pi+2k\pi,\ k\in\mathbb{Z}$

$\frac{\sin x}{\cos x} < -2 \Leftrightarrow tg\,x < -2$

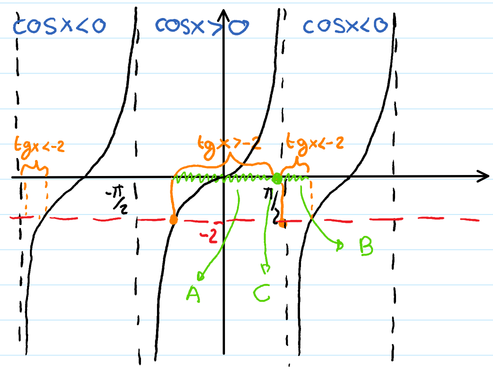
(C) if $\cos x = 0 \Leftrightarrow x = \frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$

$\sin x > 0$ $\quad \sin\left(\frac{\pi}{2}+k\pi\right)>0$ iff k is even

$x = \frac{\pi}{2}+2k\pi,\ k\in\mathbb{Z}$

$\sin x > 0$ $\qquad \sin\left(\frac{\pi}{2}+k\pi\right)>0$ iff k is even

$x = \frac{\pi}{2}+2k\pi$, $k\in\mathbb{Z}$



$$\boxed{S1: \ -\operatorname{arctg}2+2k\pi < x < \pi-\operatorname{arctg}2+2k\pi, \ k\in\mathbb{Z}}$$

(2) $2\cos x + \sin x < 1$

$2\cos x + \sin x - 1 < 0 \longrightarrow$ variable change as in EX7

$$2\,\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} - 1 < 0$$

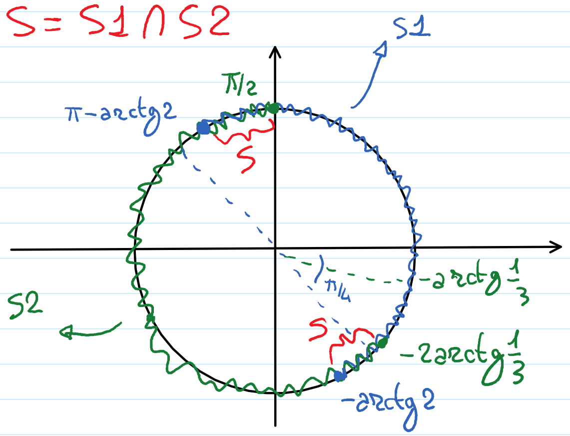
$$\frac{2-2t^2+2t-1-t^2}{1+t^2} < 0 \qquad 1+t^2>0 \ \forall t$$

$$\left[\begin{array}{l} -3t^2+2t+1<0 \qquad 3t^2-2t-1>0 \\ t_{12} = \frac{-1\pm\sqrt{1+3}}{-3} = \frac{1\mp 2}{3} = -\frac{1}{3}, 1 \end{array}\right]$$

$t<-\frac{1}{3} \vee t>1 \iff \operatorname{tg}\frac{x}{2}<-\frac{1}{3} \vee \operatorname{tg}\frac{x}{2}>1$

$$\boxed{S2: \ \pi/2+2k\pi < x < -2\operatorname{arctg}\frac{1}{3}+2(k+1)\pi, \ k\in\mathbb{Z}}$$

$\Rightarrow S = S1 \cap S2$



$$\boxed{\text{SOLUTION: } \begin{array}{l} -\operatorname{arctg}2+2k\pi<x<-2\operatorname{arctg}\frac{1}{3}+2k\pi \\ \vee \\ \frac{\pi}{2}+2k\pi<x<\pi-\operatorname{arctg}2+2k\pi \end{array}, \ k\in\mathbb{Z}}$$

EXTRA: Solve the first part with

the substitution used for
the second part.
Is it easier or more complex?

EX 9: $\dfrac{\sin x + \sqrt{3}\cos x + 1}{tg\, x - 1} \geq 0$

SOLUTION:

(N) $\sin x + \sqrt{3}\cos x + 1 \geq 0$

$$\frac{2t}{1+t^2} + \sqrt{3}\,\frac{1-t^2}{1+t^2} + 1 \geq 0 \iff \frac{2t+\sqrt{3}-\sqrt{3}t^2+1+t^2}{1+t^2} \geq 0$$

$$\frac{(1-\sqrt{3})t^2 + 2t + (1+\sqrt{3})}{1+t^2} \geq 0$$

$1+t^2 > 0 \quad \forall t$

$$t_{12} = \frac{-1 \pm \sqrt{1-(1+\sqrt{3})(1-\sqrt{3})}}{1-\sqrt{3}} = \frac{-1\pm\sqrt{3}}{1-\sqrt{3}}$$

$$= -1,\ \underbrace{-\frac{1+\sqrt{3}}{1-\sqrt{3}}}$$

$$-\frac{(1+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3}$$

$$-1 \leq t \leq 2+\sqrt{3} \iff -\frac{\pi}{4}+k\pi \leq \frac{x}{2} \leq \arctg(2+\sqrt{3})+k\pi$$

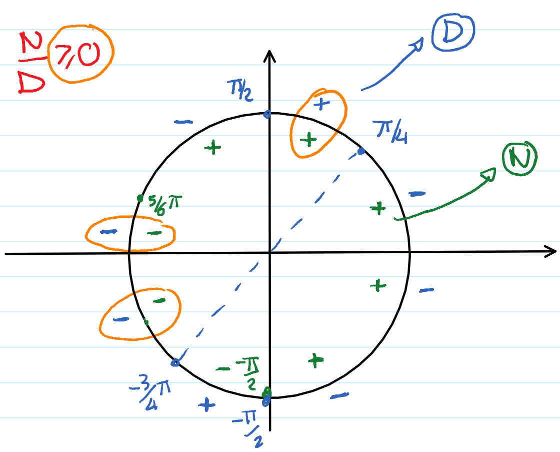
$\arctg(2+\sqrt{3}) = 75^\circ = \frac{5}{12}\pi$

N: $-\frac{\pi}{2}+2k\pi \leq x \leq \frac{5}{6}\pi + 2k\pi, \quad k\in\mathbb{Z}$

(D) $tg\, x - 1 \geq 0 \iff tg\, x \geq 1 \iff \frac{\pi}{4}+k\pi \leq x < \frac{\pi}{2}+k\pi, \ k\in\mathbb{Z}$

D: $\frac{\pi}{2}+k\pi \leq x < \frac{\pi}{2}+k\pi, \quad k\in\mathbb{Z}$

$\implies \frac{N}{D} \geq 0$



$$\boxed{\text{SOLUTION: } \frac{\pi}{4}+2k\pi < x < \frac{\pi}{2}+2k\pi \ \vee\ \frac{5}{6}\pi+2k\pi < x < \frac{3}{4}\pi+2k\pi, \quad k\in\mathbb{Z}}$$

SOLUTION : $\frac{\pi}{4} + 2k\pi < x < \frac{\pi}{2} + 2k\pi$ $\vee$ $\frac{5}{6}\pi + 2k\pi < x < \frac{3}{4}\pi + 2k\pi$, $k \in \mathbb{Z}$

EX10: $\cos 2x + \cos^2\left(\frac{x}{2}\right) \leq -\frac{1}{2}$

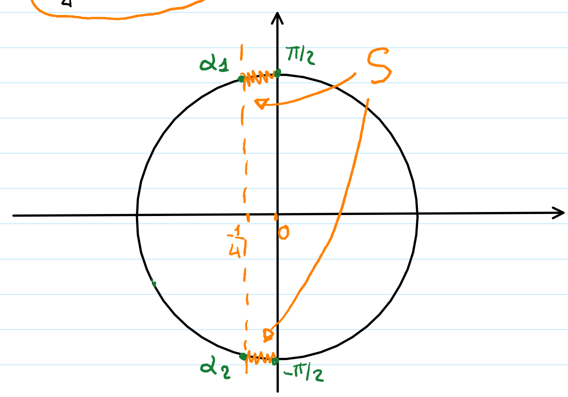
SOLUTION : Recall: $\cos 2x = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$
$\sin 2x = 2\sin x \cos x$

$\cos 2x + \cos^2\left(\frac{x}{2}\right) + \frac{1}{2} \leq 0 \iff \left(2\cos^2 x \cancel{-1}\right) + \left(\frac{1}{2}\cos x + \cancel{\frac{1}{2}}\right) + \cancel{\frac{1}{2}} \leq 0$

$4\cos^2 x + \cos x \leq 0$ with $\cos x = t$

$4t^2 + t \leq 0 \iff -\frac{1}{4} \leq t \leq 0$

S $-\frac{1}{4} \leq \cos x \leq 0$



SOLUTION : $\frac{\pi}{2} + 2k\pi \leq x \leq \alpha_1 + 2k\pi$ $\vee$ $-\alpha_2 + 2k\pi \leq x \leq -\frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$

with $\alpha_{12} = \pm\left|\pi - \arccos\frac{1}{4}\right|$