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Given a function

$f: A \longrightarrow B$

- f is called <u>injective</u>

$$\boxed{\text{if } a_1, a_2 \in A \quad a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)}$$

$$\Updownarrow$$

$$\boxed{\begin{array}{c} f(a_1) = f(a_2) \\ \Downarrow \\ a_1 = a_2 \end{array}} \quad (*)$$

Ex

$$f: \mathbb{R}^+ \longrightarrow \mathbb{R}$$

$$\mathbb{R}^+ = [0, +\infty[$$

$$f(x) = x^2 + 3$$

Is it injective?

Apply (*)

$$f(x_1) = f(x_2)$$

$$x_1^2 + \cancel{3} = x_2^2 + \cancel{3}$$

$$x_1^2 = x_2^2$$

(since $x_1 \geq 0$ $x_2 \geq 0$) $\Longrightarrow x_1 = x_2$

$$f : A \longrightarrow B$$

is sorjective

if

$\forall\, b \in B \ \exists\, a \in A$

s.t. $f(a) = b$

i.e.

$$f(A) = \{ f(a) \ a \in A \} = B$$

(in general we have
$f(A) \subseteq B$)

Remark: E, F

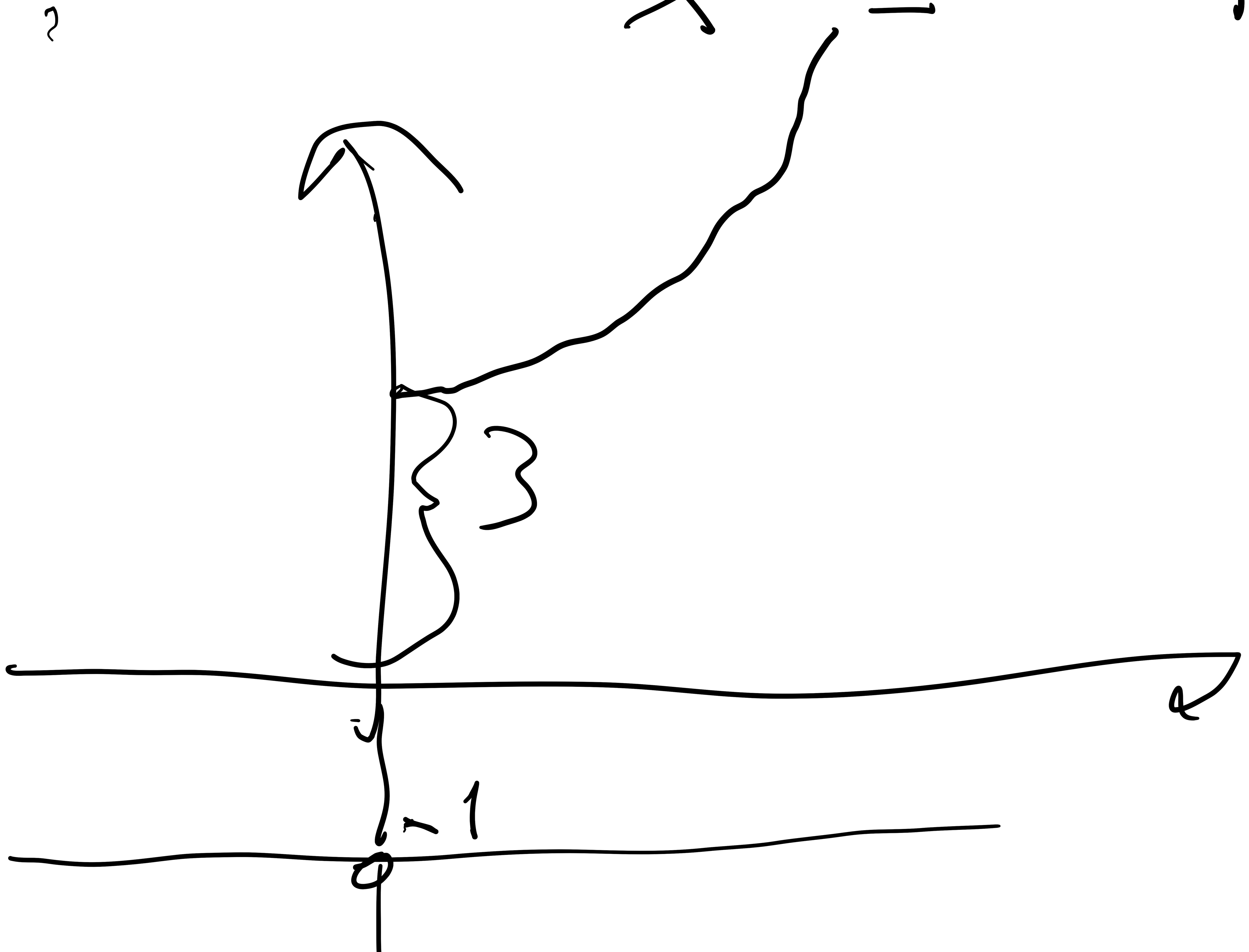
$$E = F$$
$$\Updownarrow$$
$$\underline{E \subseteq F} \text{ and } \underline{F \subseteq E}$$

Example: $f: \mathbb{R}^+ \to \mathbb{R}$

$f(x) = 3 + x^2$ surjective?

No: $-1 \in \mathbb{R}$

$$-1 = 3 + x^2$$
$$x^2 = -4 \quad \text{not possible}$$



If we change the codomain $\mathbb{R}$ with

$B = \{ y \in \mathbb{R} \quad y \geq 3 \}$

$g : \mathbb{R}^+ \longrightarrow B$ is surjective $g(x) = f(x)$

<u>Proof</u>: let $y \in B$

$$y = g(x) = 3 + x^2$$

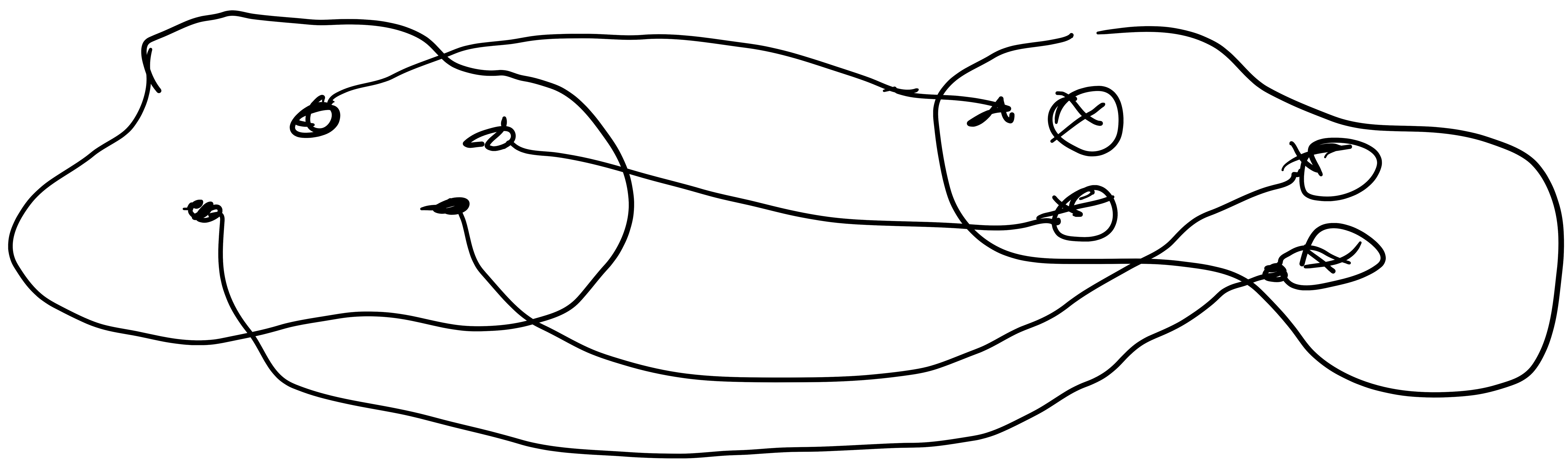
$$x^2 = 3 - y \geq 0$$

$$x = \sqrt{3 - y}$$

This $g : \mathbb{R}^+ \longrightarrow B$ is both <u>injective</u> and <u>surjective</u>

so g is bijective

Definition: If $f: A \to B$ both inj. and surj. we say f is $\boxed{\text{bijective}}$



If $f: A \longrightarrow B$ is bijective we say that there is a one to one correspondence between A and B.

$\boxed{I_1 \quad I_2 \quad \ldots \quad I_n \quad \ldots}$

$\{1\} \quad \{1,2\} \quad \{1,2 \ldots n\}$

Def A is finite if it can be put in one to one corr. with some I_n, for some $n \geq 1$, I say that "A has n elements".

Def A is infinite if it is not finite.

Examples: $\mathbb{N}$ is infinite

$O^+ = \{$Odd positive numbers$\}$

$f: \mathbb{N} \longrightarrow O^+$

$n \longmapsto 2n+1$

f is injective

$(2n_1+1 = 2n_2+1 \quad \cancel{2}n_1 = \cancel{2}n_2)$

Surjective ?

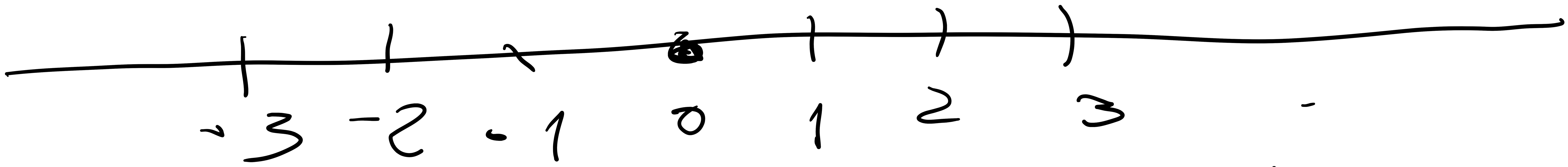
$y \in O^+$ $\quad y = 2n+1$

for some n.

$\Longrightarrow f$ is bijective

<u>Definition</u> A set is said countable if either it is finite or it is in 1-1 relationship with $\mathbb{N}$.

$$\mathbb{Z} = \{ \pm n \quad n \in \mathbb{N} \}$$



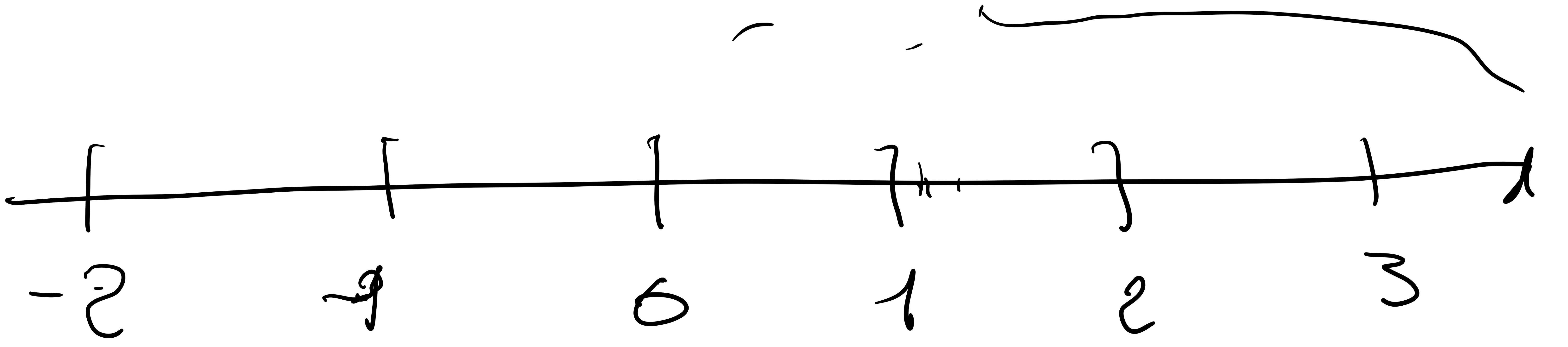
Exercise: Prove that $\mathbb{Z}$ is countable.

$\mathbb{Q} = \{ \text{rational numbers} \}$

$= \{ \text{fractions} \} = \left\{ \frac{m}{n},\ m,n \in \mathbb{Z},\ n \neq 0 \right\}$

$= \left\{ \frac{11}{100}, \frac{1593}{2022}, \dots \right\}$

$\mathbb{Q} \supset \mathbb{N} = \left\{ \frac{0}{1}, \frac{1}{1}, \frac{2}{1} \right\}$



<u>Theorem</u>: $\mathbb{Q}$ is countable

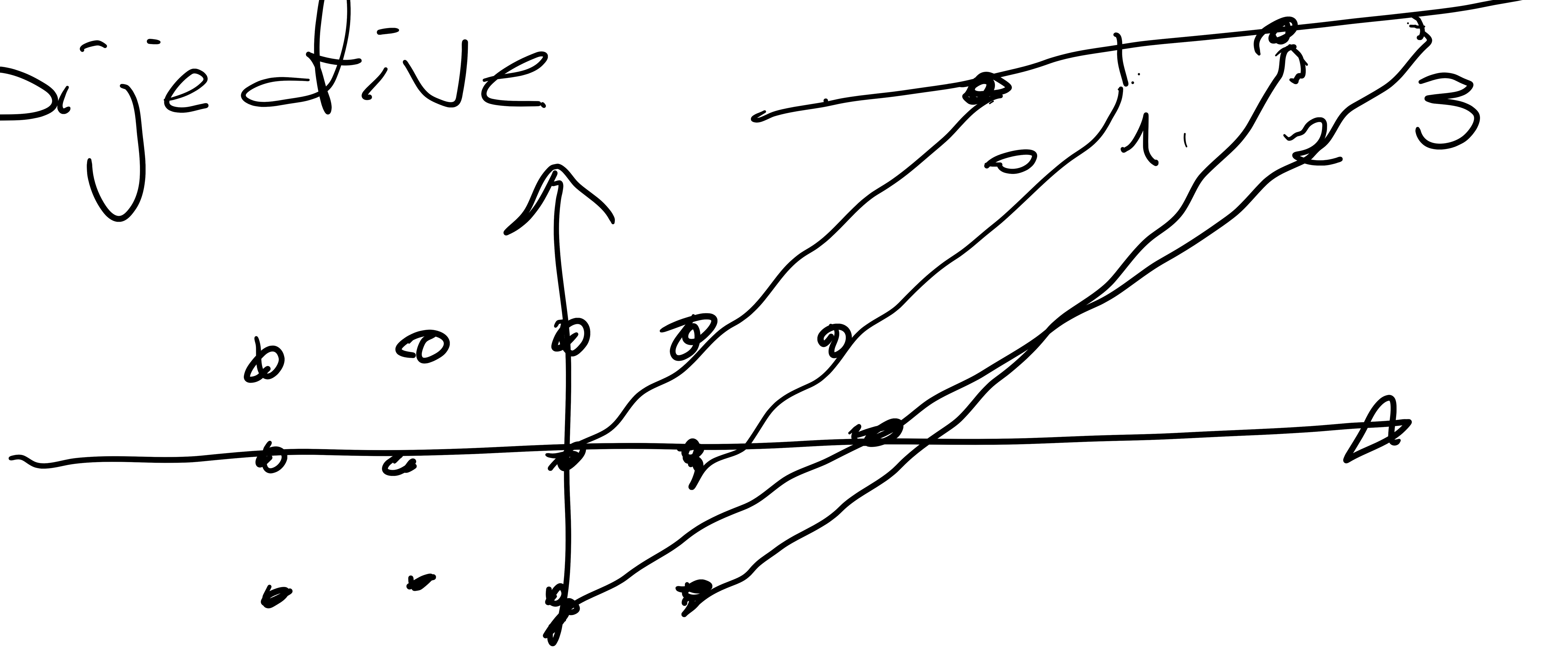
i.e. $\exists f$

$f: \mathbb{N} \longrightarrow \mathbb{Q}$

bijective

(idea

$\mathbb{Z} \times \mathbb{Z}$



A, B sets $\qquad A \times B = \{(a,b) \mid a \in A,\ b \in B\}$

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$

Pithagoras' Theorem:

$$\boxed{c^2 = a^2 + b^2}$$

Example

$a = 1 \qquad b = 1$

$$c^2 = a^2 + b^2 = 1 + 1$$
$$= 2$$

$$\boxed{c^2 = 2}$$

__Theorem__ There exist no rational numbers c such that $c^2 = 2$

$$\Updownarrow$$

$$\sqrt{2} \notin \mathbb{Q}$$

__Proof:__ By __contradiction__

$\sqrt{2} \in \mathbb{Q}$ i.e

$$\boxed{\sqrt{2} = \frac{m}{n}} \quad (*)$$

(assume that $\frac{m}{n}$ is already reduced in order that m and n have no common factors)

$(*) \implies 2 = \frac{m^2}{n^2}$

$\boxed{m^2 = 2n^2} \implies m^2$ is even (Prove: by exercise)

$\implies$ <u>m is even</u>

$m = 2q \qquad q \in \mathbb{N}$

$4q^2 = 2n^2$

$n^2 = 2q^2$

n^2 is even

n is even.

$\square$

q.e.d.

How to enlarge $\mathbb{Q}$?

$\mathbb{N} \cdots \mathbb{Z} \cdots \mathbb{Q}$

$\frac{m}{n} \in \mathbb{Q} \qquad \frac{r}{s} \in \mathbb{Q}$

$$\frac{m}{n} + \frac{r}{s} = \frac{ms + rn}{ns}$$

$$\frac{m}{n} \cdot \frac{r}{s} = \frac{mr}{ns}$$

$$\frac{m}{n} \Big/ \frac{r}{s} = \frac{m}{n} \cdot \frac{s}{r}$$

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in $\mathbb{Q}$ $\qquad q_1 = \frac{m_1}{n_1}$

$q_2 = \frac{m_2}{n_2}$

$q_1 \leq q_2$

order

Definition $A \subseteq \mathbb{Q}$

a lower bound for A is $l \in \mathbb{Q}$ s.t.

$$l \leq a \quad \forall a \in A$$

Ex $A_1 = \left\{ \frac{1}{n} \quad n \in \mathbb{N} \right\}$

$$A_2 = \left\{ \frac{5}{14}, \frac{11}{9}, \frac{3}{2} \right\}$$

a lower bound for A_1: ~~$\frac{1}{28} < \frac{1}{27}$~~

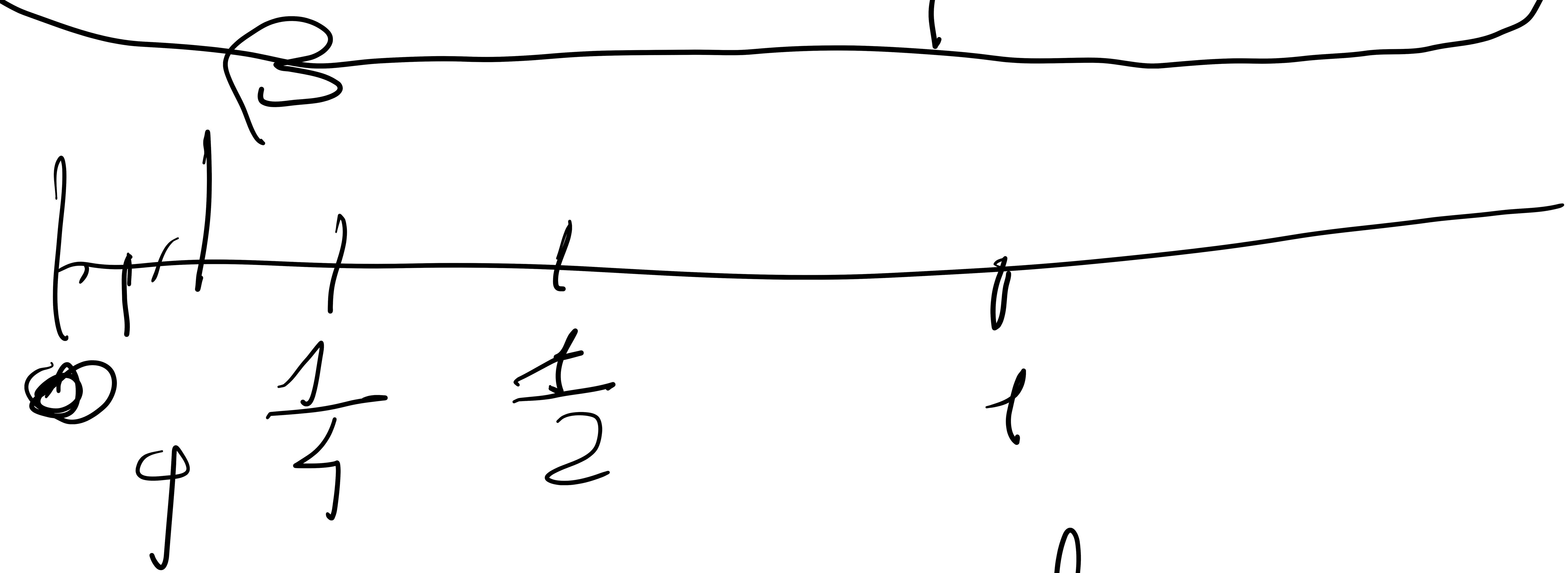
0 is a lower bound

-3 is a lower bound

The "best" lower bound is 0.

Indeed if I take $\beta \in \mathbb{Q}$, $\beta > 0$ I can find

(P) $q \in A_1 \quad 0 \leq q \leq \beta$



0 is the only one satisfying (P)

<u>Definition</u> the "<u>best lower bound</u>" is a lower bound that satisfies (P)

We call the best lower bound the $\boxed{\text{INFIMUM of } A_1}$