There have been some problems with the writing program this morning, so there is not a recording of my lecture at the black board. So let me summarize here the contents of today's lecture.

$P \Rightarrow Q$

- "P implies Q" means
- "if P is true then Q is true"
- "Q if P" ... "P is sufficient for Q"
- "P only if Q" "Q is necessary for P"

P = it is raining
Q = the streets are wet

- $P \Rightarrow Q$
- P is a suffient condition for Q
- Q is a necessary condition for P

Remark 1: $P \Rightarrow Q$ does not mean that $Q \Rightarrow P$.
(streets can be wet for reasons other than raining)

Remark 2 $\boxed{P \Rightarrow Q}$ is equivalent to $\boxed{\text{non } Q \Rightarrow \text{non } P}$

Indeed, if non $\mathcal{P}$ were false, then
$\mathcal{P}$ is true $\Rightarrow$ Q is true

$\boxed{\text{non } Q \Rightarrow \text{non } \mathcal{P}} \Rightarrow \text{non}(\text{non}\,\mathcal{P}) \Rightarrow \text{non}(\text{non}\,Q)$

$\mathcal{P} \Rightarrow Q$

$$\boxed{\mathcal{P} \Rightarrow Q} \Longleftrightarrow \boxed{\text{non}\,Q \Rightarrow \text{non}\,\mathcal{P}}$$

"$\Leftrightarrow$" means "$\Rightarrow$" "$\Leftarrow$"

- $\mathcal{A} \Leftrightarrow \mathcal{B}$
  is the same as
- $\mathcal{A}$ if and only $\mathcal{B}$
  is the ~~same as~~
- $\mathcal{A}$ is a necessary and sufficient condition for $\mathcal{B}$
- $\mathcal{A}$ is equivalent to $\mathcal{B}$

Example
A poligon has 3 edges
$\Updownarrow$
A poligon has 3 angles

SETS.
A set is a collection of objects which are called elements.

A is a set.

$a \in A$ means
a is an element of A
"
a belongs to A

$A = \{\text{poligons}\}$

$\triangle \in A$

$\bigcirc \notin A$ ...... $\bigcirc$ doesn't belong to A
"
$\bigcirc$ is not an element of A.

B is a set, A is a set
$B \subseteq A$ "B is contained in A" "B is included in A"
"
every element of B is also an element of A.

$B = \{\text{people from Tanzania}\}$
$A = \{\text{people from Africa}\}$
$B \subseteq A$

E, F are subsets of A

$E \cap F$ "intersection of E and F"

"the set of elements which are both in E and in F"

$\mathbb{N} = \{\text{natural numbers}\} = \{0, 1, 2, 3, \dots\}$

$E = \{ n \in \mathbb{N} \text{ such that there exists } m \in \mathbb{N} \text{ verifying } n = 2m \}$

$F = \{ n \in \mathbb{N} \text{ s.t. there exist } m \in \mathbb{N} \ \dots \ n = 3m \}$

$E \cap F = \{ n \in \mathbb{N} \text{ which have both } 2 \text{ and } 3 \text{ as factors} \} =$

$= \{0, 6, 12, 18 \dots\dots\}$

$E, F \subseteq A$

$E \cup F = \{ c \in A \,.\, c \in E \text{ or } c \in F \}$

E and F are as above

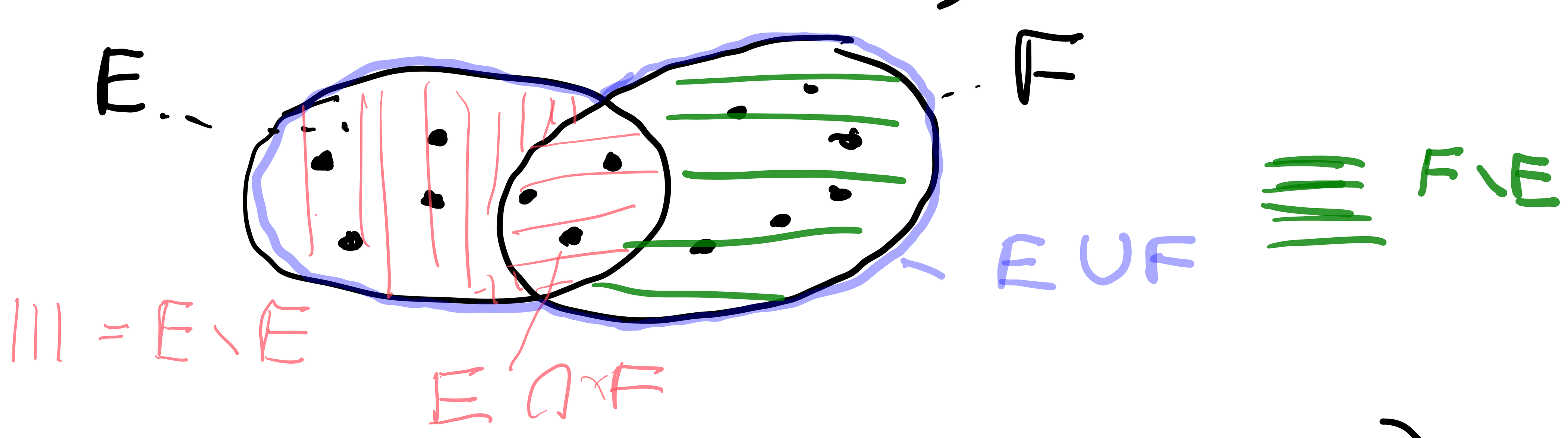
$E \cup F = \{0, 2, 3, 4, 6, 8, 9, 10 \dots\}$

Remark: $\boxed{A \text{ or } B \text{ means}}$

1) A is true but B is not
2) B is true but A is not
3) A and B are both true

instead $\boxed{\text{either } A \text{ or } B \text{ means}}$

only 1) or 2) can occur.



$\setminus$ $\quad E \setminus F = \{ c \text{ s.t. } c \in E,\ c \notin F \}$

Remark $E \cap F = F \cap E$

$E \cup F = F \cup E$

$E \setminus F \neq F \setminus E$

$\emptyset$ denotes the "empty set"

i.e. $\emptyset$ the set with no elements

$E = \{\text{even natural numbers}\} \setminus \{0\}$

$O = \{\text{odd} \quad " \quad "\} \setminus \{0\}$

$E \cap O = \emptyset$

"Function" $f: A \longrightarrow B$

where

A is a set, called the DOMAIN

B is a set, called the CODOMAIN (also RANGE)

f is the rule that

"for every" $\forall a \in A \quad f(a) \in B$

$a \longmapsto f(a)$

"f maps a into $f(a)$"

$f(a)$ is called "the image of a trough f"

$f: \mathbb{N} \longrightarrow \mathbb{N}$

$n \longmapsto f(n) = 2n$

~~$g: \mathbb{N} \longrightarrow \mathbb{Q} = \{\text{fractions } \frac{r}{s}\}$~~

~~$n \longmapsto f(n) = \frac{n^3 + 2n}{n-2}$~~ ; $n \neq 2$!

$g: \mathbb{N} \setminus \{2\} \longrightarrow \mathbb{Q}$

$n \longmapsto f(n) = \frac{n^3 + 2n}{n-2}$

Definition: A function $f: A \to B$ is said INJECTIVE if $\forall a_1, a_2 \in A$

$a_1 \neq a_2 \Longrightarrow f(a_1) \neq f(a_2)$

Examples: $\mathbb{Z} = \{\text{integer numbers}\}$

$\{\text{natural numbers with a sign + or −}\}$

$= \{0, -1, 1, -2, 2, -3, 3, \ldots\}$

$f: \mathbb{Z} \longrightarrow \mathbb{N}$ is f injective?

$n \longmapsto f(n) = n^2$

No: $n_1 = -3 \quad n_2 = 3$

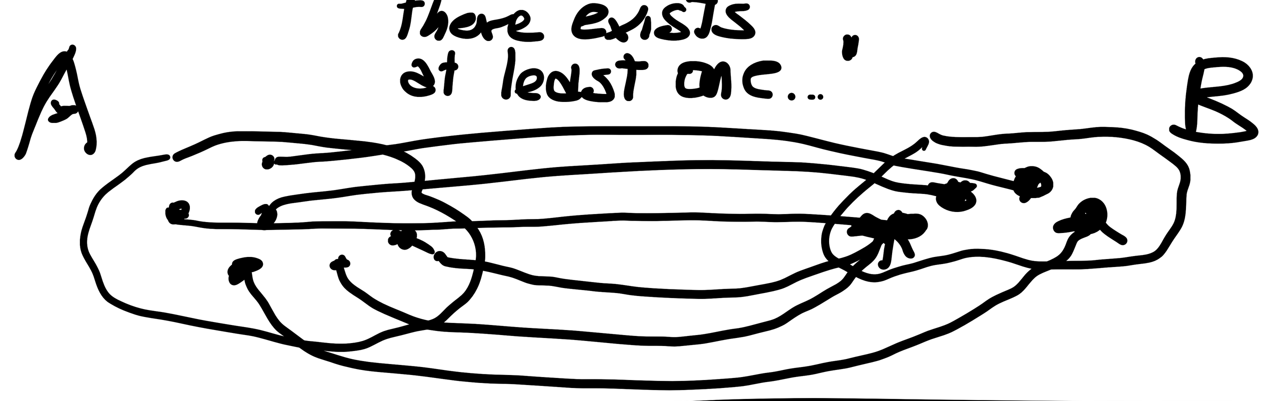
$f(n_1) = (-3)^2 = 9$

$f(n_2) = (3)^2 = 9$

Definition: A function $f: A \longrightarrow B$ is said surjective if

$\forall b \in B \; \exists a \in A \quad b = f(a)$

"there exists at least one..."



$f: \mathbb{Z} \longrightarrow \mathbb{N}$ is it surjective?

$n \longmapsto n^2$

No: for example $8 \in \mathbb{N}$ is not the image of any $n \in \mathbb{N}$: indeed $f(n) = n^2 = 8$ has no solutions in the domain $\mathbb{Z}$

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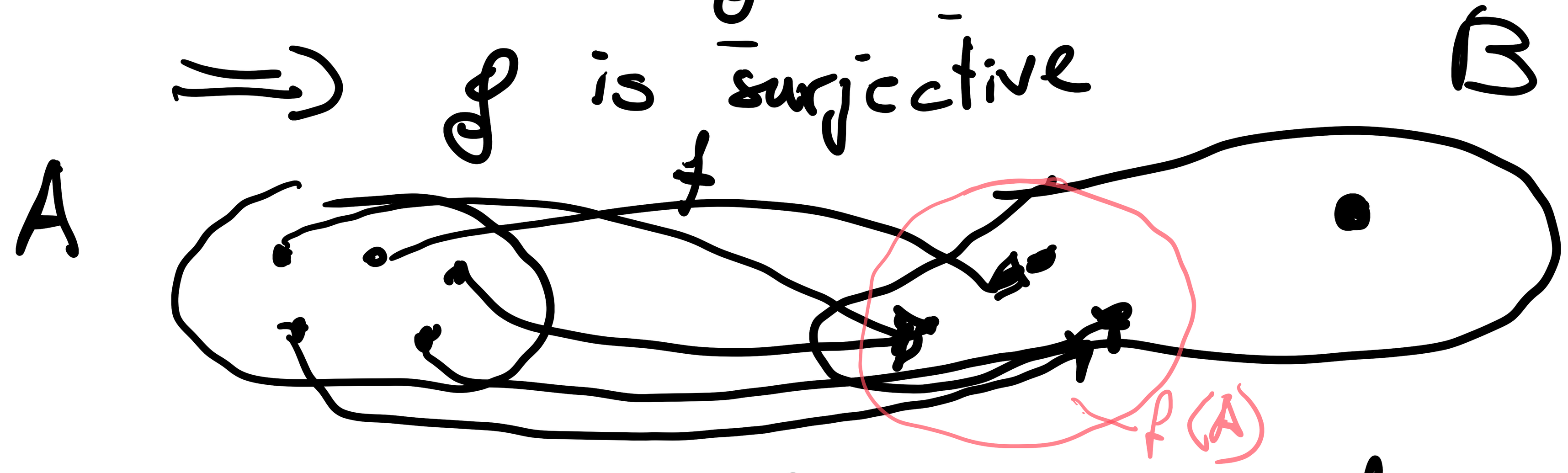
1) Let us observe that f is <u>surjective</u> $\underbrace{\text{if and only if}}_{\text{iff}}$ $f(A) = \{f(a),\ a \in A\}$ "$= B$".

(In general, $f(A) \subseteq B$)

2) From <u>any</u> function $f: A \to B$ $\supset$ can construct

$$g: A \to \boxed{\tilde{B} = f(A)}$$
$$a \mapsto g(a) = f(a)$$

$\Longrightarrow$ g is surjective



g is the same as f except that we take a smaller codomain

$$f: \mathbb{Z} \to \mathbb{N}$$
$$n \mapsto n^2$$

$$\tilde{B} = f(\mathbb{Z}) = \{0, 1, 4, 9, 16, 25, \ldots\ldots\}$$

$$g: \mathbb{Z} \to \tilde{B}$$
$$n \mapsto f(n) = f(n) = n^2$$

is surjective

$$h: \mathbb{Z} \to \mathbb{Z}$$
$$n \mapsto h(n) = n^3$$

$$h(n_1) = n_1^3 = n_2^3 = f(n_2)$$
$$\Updownarrow$$
$$n_1 = n_2$$

h is injective

h is not surjective, but if we modify the codomain

$$k: \mathbb{Z} \to \{0, \pm 1, \pm 8, \pm 27, \ldots\}$$

k is (injective and) surjective