# Calculus I

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# Contents

CHAPTER 1

# Basic concepts

## 1.1. Some logics

While a formal introduction to Logics is far beyond the scope of these notes, in this chapter we provide the student with some notions which happen to be crucial for the rigorous exposition of the material of these notes.

If $\mathcal{P}$ and $\mathcal{Q}$ are propositions, the notation

$$\mathcal{P} \implies \mathcal{Q} \tag{1.1.1}$$

means that *if* $\mathcal{P}$ *is true than also* $\mathcal{Q}$ *is true*. This can be expressed also by saying that *"$\mathcal{P}$ implies $\mathcal{Q}$"*, or, equivalently, *"$\mathcal{P}$ is a sufficient condition for $\mathcal{Q}$"*, or *"$\mathcal{Q}$ if $\mathcal{P}$"*, or *"$\mathcal{Q}$ is a necessary condition for $\mathcal{P}$"*, or *"$\mathcal{P}$ only if $\mathcal{Q}$"*.

For instance, if $\mathcal{P}$ is "It rains" and $\mathcal{Q}$ is "The streets are wet", one has $\mathcal{P} \implies \mathcal{Q}$ which means

"the fact that it is raining implies that the streets are wet", which can be *equivalently* expressed by one of the sentences:

"the fact that it rains is a sufficient condition for the streets to be wet";

"the streets are wet if it rains";

"the fact that the streets are wet is a necessary condition for it rains";

"it rains only if the streets are wet".

From this same example one deduce that, in general, $\mathcal{P} \implies \mathcal{Q}$ **does not imply** $\mathcal{Q} \implies \mathcal{P}$, (because the streets can be wet for other reasons.)

However,

$$(\mathcal{P} \implies \mathcal{Q}) \quad \implies \quad (\text{non}\mathcal{Q} \implies \text{non}\mathcal{P}), \tag{1.1.2}$$

whose proof is as follows: if non$\mathcal{Q}$ holds true, then $\mathcal{P}$ cannot be true, otherwise, by $(\mathcal{P} \implies \mathcal{Q})$, we would get that $\mathcal{Q}$ would be true as well...but this is false because we have began with the assumptio that non$\mathcal{Q}$ holds true (and $\mathcal{Q}$ and non$\mathcal{Q}$ cannot be true at the same time.) By using the fact that double negation coincides with affirmation, from (1.1.2) we get also

$$(\text{non}\mathcal{Q} \implies \text{non}\mathcal{P}) \quad \implies (\text{non-non}\mathcal{P} \implies \text{non-non}\mathcal{Q}), \text{ i.e. } (\mathcal{P} \implies \mathcal{Q}).$$

Summarizing the two above implications, we have, $(\mathcal{P} \implies \mathcal{Q})$ is *equivalent* to $(\text{non}\mathcal{Q} \implies \text{non}\mathcal{P})$, which can be written as

$$\boxed{(\mathcal{P} \implies \mathcal{Q}) \quad \iff \quad (\text{non}\mathcal{Q} \implies \text{non}\mathcal{P})}$$

Notive that we have utilized the notation $\iff$, which reads *if and only if* . Therefore the proposition

$$\mathcal{A} \iff \mathcal{B}$$

means $\mathcal{A}$ *is true if and only if* $\mathcal{B}$ *is true*.It can be equivalently expressed saying $\mathcal{A}$ *is a necessary and sufficient condition for* $\mathcal{B}$ or even $\mathcal{A}$ *is equivalent to* $\mathcal{B}$.

For instance, if $\mathcal{A}$ is the proposition "$\Lambda$ is a square" and $\mathcal{B}$ is the proposition "$\Lambda$ has 4 edges", we have that $\mathcal{A} \implies \mathcal{B}$ is true, while $\mathcal{B} \implies \mathcal{A}$ is false (since there exist polygons with 4 edges which are not squares). As a consequence, also $\mathcal{A} \iff \mathcal{B}$ is false. Instead, if $\mathcal{C}$ = "$\Lambda$ is a polygon with 4 equal angles and such that consecutive edges are congruent", then $\mathcal{A} \iff \mathcal{C}$ is true.

Much care has to be put into the use of the conjunctions *and* and *or*. If we say "$\mathcal{P}_1$ and $\mathcal{P}_2$" we are saying that both $\mathcal{P}_1$ and $\mathcal{P}_2$ hold true, while if we say "$\mathcal{P}_1$ or $\mathcal{P}_2$" the following three cases may occur: i) both $\mathcal{P}_1$ and $\mathcal{P}_2$ hold true, ii) $\mathcal{P}_1$ is true but $\mathcal{P}_2$ is false, iii) $\mathcal{P}_2$ is true but $\mathcal{P}_1$ is false.

Be also careful that "$\mathcal{P}_1$ or $\mathcal{P}_2$" is different from "either $\mathcal{P}_1$ or $\mathcal{P}_2$": indeed, the latter rules out the fact that both $\mathcal{P}_1$ and $\mathcal{P}_2$ hold true.

EXAMPLE 1.1.1. $\mathcal{P}$*: Jack is from UK and his mother is from Sudan*
$\mathcal{Q}$*: Jack is from UK or his mother is from Sudan*
$\mathcal{R}$*: Either Jack is from UK or his mother is from Sudan*

In constructing propositions, we will use the notation $\exists$ with the meaning of "there exists one", or, equivalently, "there exists at least one", while the notation $\forall$ means "for every". $\exists$ is called the *existence quantifier* and $\forall$ is called the *universal quantifier.* For instance,

*"the sum of the angles of $T$ is $\pi$, $\forall$ triangle $T$* means that for every triangle the sum of the angles is $\pi$. Instead, $\exists$ *polygons such that the sum of their angles is* $2\pi$ means that the sum of the angles for some polygons (the quadrangles!) is $2\pi$. [1]

## 1.2. Sets

**Definition 1.2.1**

We call **set** any collection of objects, which are denominated **elements** .

A set $A$ is given as soon we describe its objects. For instance,

$$A = \{b, \gamma, casa, stella, H\}$$

denotes the set $A$ whose elements are the letters $b, \gamma, H$, and the words $casa$, $stella$. To say that an object is an element of a set one uses the notation $\in$, while we use $\notin$ to mean that an object is not an element of a certain set: for instance, for the previous set $A$ one has $\gamma \in A$, $\pi \notin A$.

For instance, in the previous example we have $\gamma \in A$ ($\gamma$ *belongs to* $A$) and also $casa \in A$ ($casa$ *belongs to* $A$).

**Definition 1.2.2**

A set $B$ **is a subset of** $A$, and we write $B \subseteq A$, if any element of $B$ is also an element of $A$. In this case, one also says that $A$ **contains** $B$, or $B$ **is contained** $A$.

[1]It is well-known that for a polygon with $\ell$ ($\ell \geq 3$) edges the sum of the internal angles is $(\ell - 2)\pi$. For instance, for quadrangles, pentagons, and hexagons, this sum is $2\pi, 3\pi, 4\pi$, respectively.

For instance, if $A = \{b, \gamma, casa, stella, h\}$, $B = \{b, stella\}$, and $K = \{\gamma, stella, h\}$, then

$$B \subseteq A \qquad \text{and} \qquad K \subseteq A$$

Of course, for any set $E$, it is $E \subseteq E$.

If $E$ and $F$ are sets we write $F \subset E$ to mean that $F \subseteq E$ and $F \neq E$. However, to stress the fact that $F$ is not equal to $E$, it is common use to use the notation $F \subsetneq E$ instead of the mere $F \subset E$.

If an environment set $A$ is given, one can also describe a subset $E \subseteq A$ by means of a property of its.[2]. For instance, if $A = \mathbb{N}$ is the set of natural numbers, the subset $E \subset A$ of even numbers can be descrbed as

$$E = \{0, 2, 4, 6, 8.....\}$$

or, more rigorously, as

$$E = \{2m, \ \ m \in \mathbb{N}\}.$$

Example 1.2.3. If $P = \{\text{members of the Italian Parliament on September 27, 2022}\}$, one can consider the subset

$$M = \{a \in P, \text{the first name of } a \text{ is Matteo}\}.$$

For sure, we have that

$$\text{Matteo Salvini} \in M, \qquad \text{Matteo Renzi} \in M,$$

or, equivalently

$$\{\text{Matteo Salvini, Matteo Renzi}\} \subseteq M \subseteq P.$$

Probably, it is also $\{\text{Matteo Salvini, Matteo Renzi}\} \subset M$ (because there might be other members of the Parliament whose first name is Matteo). And it is also $M \subset P$ (because, for instance, the first name of Maria Elena Boschi is Maria Elena and not Matteo).

For any fixed number $n \geq 1$, a subset of $P$ is given by

$$C_n = \{a \in P, \text{the age of } a \text{ is larger than } n \text{ on September 27,2020}\}$$

One has Maria Elena Boschi$\notin C_{43}$, while Silvio Berlusconi$\in C_{83}$. And, of course, $C_{120} = \emptyset$

In the previous example, we have used the notation $\emptyset$, which means **empty set** , and is defined as *the set with zero elements*. Clearly the empty set is a subset of *every* set. There are infinitely many alternative ways to describe the empty set. For instance,

$$\emptyset = \{\text{the traffic lights that play football}\} = \{\text{the cows that know Pythagoras'theorem }\} =$$

$$= \{\text{the factors of 11 different from 11 }\} = \{\text{the hexagons such that the sum of the internal angles is } 6\pi \ \}$$

If $E, F$ are sets, their **intersection** $E \cap F$ and their **union** $E \cup F$ are defined as

$$E \cap F = \{a, \ \ a \in E \ \text{ and } \ a \in F\}$$

and

$$E \cup F = \{a, \ \ a \in E \ \text{ or } \ a \in F\}$$ [3]

[2]Be careful that the fact that $E$ is subset of a given set $A$ plays a crucial role here, as is shown by Russel's paradox, see e.g. https://en.wikipedia.org/wiki/Russell%27$s$_paradox

The **difference** $E\backslash F$ **of** $E$ **and** $F$ is defined as

$$E\backslash F = \{a \in E, \ \ a \notin F\}$$

Notice that, for all sets $E, F$, one has $E \cup F = F \cup E$ $F \cap F = F \cap E$, while, in general $E\backslash F$ is different from $F\backslash E$

EXAMPLE 1.2.4. If $M, C$ are as in Example 1.2.3 one has
$M \cap C = \{$elements of $P$ who are strictly older than 40 and whose first name is Matteo$\}$
$M \cup C = \{$elements of $P$ who are strictly older than 40 or whose first name is Matteo$\}$
$M\backslash C = \{$ elements of $P$ who are younger than 40 and whose first name is Matteo$\}$ $C\backslash M = \{$elements of $P$ who are stri

If $P_{Fr} = \{$members of the French Parliament$\}$ then
$C \cap P_{Fr} = P \cap P_{Fr} = P_{Fr} \cap \mathbb{N} = \emptyset$

EXAMPLE 1.2.5. If $G = \{n \in \mathbb{N}, \ \ \exists m \in \mathbb{N} \ \ n = 3m\}$, and $H = \{n \in \mathbb{N}, \ \ \exists m \in \mathbb{N} \ \ n = 4m\}$ then
$G \subset \mathbb{N}, \quad H \subset \mathbb{N}$
$G \cap H = \{n \in \mathbb{N}, \ \ \exists m \in \mathbb{N} \ \ n = 12m\}$
$G \cup H = \{n \in \mathbb{N}, \ \ \exists m \in \mathbb{N} \ \ n = 3m \text{ or } n = 4m\}$
$G\backslash H = \{n \in \mathbb{N}, \ \ \exists m \in \mathbb{N} \ \ n = 3m \text{ and } n \neq 4\ell \ \forall \ell \in \mathbb{N}\}$
$G\backslash \mathbb{N} = \emptyset$,

If we are given two sets $X, Y$, the **product set** $X \times Y$ is defined as the set of pairs $(x, y)$ such that $x \in X$ and $y \in Y$, namely

$$X \times Y = \{(x, y), \quad x \in X, y \in Y\}$$

For instance, if $G, H$ are as above

$$G \times H = \{(h, g), \quad \exists m, \ell \in \mathbb{N}, h = 3m, g = 4m\}$$

In particular $(720, 720) \in G \times H$ and $(9^4, 10^2) \in G \times H$, while $(720, 721) \notin G \times H$ and $(9^4, 10) \notin G \times H$.

Observe that $X \times Y \neq Y \times X$, unless $X = Y$

More in general, for a finite number of sets $X_1, X_2, \ldots, X_n$, the product $X_1 \times X_2 \times \ldots \times X_n$ is defined as the set of (ordered) $n$-tuples $(x_1, x_2, \ldots, x_n)$ verifying $x_i \in X_i$, $\forall i$ such that $1 \leq i \leq n$.

## 1.3. Theorems

Mathematics is expressed through *definitions* and *theorems*. What is the *statement of a theorem*? Actually, it is a very simple object: the statement of a theorem reads as

**Theorem.** *If* $\mathcal{I}$ *is true than also* $\mathcal{T}$ *is true.*

Here $\mathcal{I}$ and $\mathcal{T}$ are propositions, called *hypothesis* and *thesis*, respectively. The *proof* of a theorem consists in passing from the hypothesis to the thesis passing through implication steps.

For example, suppose we have the theorem

[3]Be careful with the meaning of the conjunction *or* (see above) .

**Theorem.** *The square of the diagonal of a quadrangle having all equal angles is equal to the sum of the squares of two consecutive edges.*

Proof. A quadrangle having all equal angles is a rectangle $\implies$ the diagonal $AC$ of a rectangle $ABCD$ ($A, B, C, D$ are consecutive vertexes ) is nothing but the hypotenuse of the right triangle $ABD$: apply Pithagoras Theorem to $ABD \implies \implies$ Thesis. □

The words **Lemma, Proposition, Corollary** are synonymous of **Theorem**.

**1.3.1. Kinds of proof.** There is not something like *a general method for proving a theorem*, and, as said above, a proof is a chain of implications between the hypothesis and thesis. We will see several theorem's proofs in this course, each of which requires some intuitive ideas to be made formal eventually. However, among the various kinds of proof, there are two *methods* that sometimes result very useful:*proof by induction* and the *proof by contradiction*.

**The proof "by contradiction".** Suppose you have to prove the theorem: *if the hypothesis $\mathcal{I}$ holds then the thesis $\mathcal{T}$ holds.* The proof by contradiction consists in denying the thesis, and then proceeding by some implications until one reaches a contradiction with the hypotesis. S, the proof by contradiction always it starts with "If (by contradiction) $\mathcal{T}$ were false...", and proceeds by some implications until one reaches a statement $\mathcal{C}$"which contradicts the hypothesis $\mathcal{I}$. [4]

**The proof "by induction"**: this method applies to theorems whose statement has the following form

Theorem 1.3.1. $\forall n \in \mathbb{N} \quad \mathcal{P}_n$ *holds true.*

The proof by induction consists in two steps:

(1) Prove that $\mathcal{P}_1$ is true;
(2) Prove that, for any number $n$, if $\mathcal{P}_n$ is true, then also $\mathcal{P}_{n+1}$

Once one has proven both 1) and 2), the theorem is proved,[5] i.e. $\mathcal{P}_n$ holds true for any $n \geq 1$.[6]

## 1.4. Functions

Let us introduce the notion of *function* in a very general setting, though in these notes we will limit to functions whose domain and codomain $B$ are subsets of the set $\mathbb{R}$ of real numbers.[7]

A *function*, also called *map*, is an object consisting of a set $A$ called *domain*, a set $B$ called *codomain*, and a *law* $f$ that, to every element of $a \in A$, associates one (and only one) element $f(a) \in B$. This is written as

$$\begin{aligned} f: \quad & A \to B \\ & a \mapsto f(a) \quad \forall a \in A \end{aligned}$$

[4] $\mathcal{C}$ contradicting the hypothesis $\mathcal{I}$ means that $\mathcal{C}$ and $\mathcal{I}$ cannot be both true.

[5] Why this method is working is something one can find in any Logics book, or even on https://en.wikipedia.org/wiki/Proof_by_contradiction♯Justification.

[6] One can replace the first step with " Prove that $\mathcal{P}_k$ is true"', for some $k \in \mathbb{N}$", and the second step with" if $\mathcal{P}_n$ is true, then also $\mathcal{P}_{n+1}$ for $n \geq k$. This is enough in order to prove that $\mathcal{P}_n$ is true for every $n \geq k$.

[7] See the next Section for a rigorous definition of real number.

One says that *"$f$ maps $a$ to $f(a)$"*, or also that *"$f(a)$ is the image of $a$"*. Moreover, for any $E \subseteq A$ the set

$$f(E) = \{f(a), \quad a \in E\} \subset B$$

is called *the $f$-image of $E$*.

For a given map $f : A \to B$ and for any $K \subseteq B$, the set

$$f^{\leftarrow}(K) = \{a, \quad f(a) \in K\} \qquad [8]$$

is called the *$f$-preimage of $K$*, or also the *the inverse $f$-image of $K$* (When the map $f$ is understood from the context, one simply uses the words *image*, *preimage*, and *inverse image* instead of $f$-image, $f$-preimage, and *inverse $f$-image*, respectively ) .[9]

Let us see a few examples:

EXAMPLE 1.4.1. *C*onsider the domain $A = \mathbb{N}$ and the codomain $B = \{n + 100, \ \ n \in \mathbb{N}\}$ and the map $f : A \to B$ defined by setting $f(m) = m^2 + 200$, $\forall m \in \mathbb{N}$. For instance,

$$f(2) = 204, \quad f(15) = 425, \quad f(0) = 200, \quad f(100) = 10200.$$

and

$$f(\{2, 3, 7\}) = \{204, 209, 249\}, \qquad f^{\leftarrow}\Big(\{n \in \mathbb{N}, \ \ 250 < n \leq 300\}\Big) = \{8, 9, 10\}$$

$$f^{\leftarrow}\Big(\{n \in \mathbb{N}, \ \ 180 < n \leq 199\}\Big) = \emptyset$$

EXAMPLE 1.4.2. Consider the domain $A = \mathbb{N} \times \mathbb{N}$ and the codomain $B = \{n - 100, \ \ n \in \mathbb{N}\}$ and the map $f : A \to B$ defined by setting $f(n_1, n_2) = n_1 n_2 - 50$,[10] $\forall (n_1, n_2) \in \mathbb{N} \times \mathbb{N}$. The map is well defined, because $f(n_1, n_2) \geq -50 \ \forall (n_1, n_2) \in \mathbb{N} \times \mathbb{N}$. For instance,

$$f(1, 0) = -50, \quad f(2, 3) = -44, \quad f(10, 2) = -30, \quad f(101, 60) = 6010.$$

$$f(\{(2, 3), (3, 8), (7, 10)\}) = \{-44, -26, 20\}, \quad f(A) = f(\mathbb{N} \times \mathbb{N}) = \{m - 50, \ \ m \in \mathbb{N}\} \ (\subset B)$$

Let us prove the last equality. One has $f(\mathbb{N} \times \mathbb{N}) \subseteq \{m - 50, \ \ m \in \mathbb{N}\}$. Indeed an element of $f(\mathbb{N} \times \mathbb{N})$ has form $n_1 n_2 - 50$, and $n_1 n_2 - 50 \in \{m - 50, \ \ m \in \mathbb{N}\}$, since $n_1 n_2 \in \mathbb{N}$.

Conversely, let us prove the inclusion

$$f(\mathbb{N} \times \mathbb{N}) \supseteq \{m - 50, \ \ m \in \mathbb{N}\}.$$

If $q \in \{m - 50, \ \ m \in \mathbb{N}\}$, it means that $q = \tilde{m} - 50$ for some $\tilde{m} \in \mathbb{N}$. On the other hand, one has $\tilde{m} = \tilde{m} \cdot 1$, so that $q = f(\tilde{m}, 1)$, i.e. $q \in f(\mathbb{N} \times \mathbb{N})$, so the inclusion is proved.

Hence we have proved that both $f(\mathbb{N} \times \mathbb{N}) \subseteq \{m - 50, \ \ m \in \mathbb{N}\}$ and $f(\mathbb{N} \times \mathbb{N}) \supseteq \{m - 50, \ \ m \in \mathbb{N}\}$, thence $f(\mathbb{N} \times \mathbb{N}) = \{m - 50, \ \ m \in \mathbb{N}\}$.

One also verifies that

$$f^{\leftarrow}\Big(\{n \in \mathbb{N}, \ \ 250 < n \leq 300\}\Big) = \{8, 9, 10\}$$

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[8]Sometimes the notation $f^{-1}(K)$ is used instead of $f^{\leftarrow}(K)$. Yet, this notation is a bit ambiguous, since it might erroneously suggest that there exists an *inverse function* $f^{-1}$ (see below in these notes), which in general is not true.

[9]Observe that $f^{\leftarrow}(K) = \emptyset$ as soon as $K \cap f(A) = \emptyset$

[10]To be rigorous, one should write $f((n_1, n_2))$, but it is customary to use the simpler notation $f(n_1, n_2)$ instead.

$$f^{\leftarrow}\Big(\{n \in \mathbb{N},\ \ 180 < n \leq 199\}\Big) = \emptyset$$

Example 1.4.3. $A = \mathbb{R}$, $B = [-2, 2]$, where $\mathbb{R}$ denotes the set of real numbers and $[-1, 1]$ is the closed interval of extremes $-1, 1$ (see the next chapter). We consider the function set $f : \mathbb{R} \longrightarrow [0, 2]$ defined as $f(x) = \sin^2 x$ (let us recall that $\sin x$ is the ordinate of the point on the circle of radius 1 corresponding to an angle of $x$ radiants). So, in particular

$$f(0) = 0, \quad f(\pi/2) = f(-\pi/2) = f(101\pi/2) = 1, \quad f(-\pi/3) = 3/4.$$

Furthermore,

$$f(\mathbb{R}) = f([\pi/2, \pi/2]) = f([0, \pi/2]) = [0, 1]$$

**1.4.1. Composition of maps.** One can *compose* maps:

**Definition 1.4.4**

If $f : A \longrightarrow B$ and $g : C \longrightarrow D$ are maps, with $B \subseteq C$, one can define the map $g \circ f : A \longrightarrow D$ by setting, for every $x \in A$,

$$g \circ f(x) = g\big(f(x)\big)$$

The map the map $g \circ f : A \longrightarrow D$ is called *the composition of $g$ with $f$*, or $g$ *after* $f$.

For example, if

$$f : \mathbb{N} \longrightarrow \{q \in \mathbb{Q}, q \leq 2\}, \quad f(n) := 1 + \frac{1}{n+1}, \qquad g : \mathbb{Q} \longrightarrow \mathbb{Q}, \quad g(q) := 8q^2$$

we can construct $g \circ f : \mathbb{N} \longrightarrow \mathbb{Q}$ (because $\{q \in \mathbb{Q}, q \leq 2\} \subset \mathbb{Q}$). It turns iout that

$$g \circ f(n) = 8\left(1 + \frac{1}{n+1}\right)^2, \qquad \forall n \in \mathbb{N}$$

We will see several examples of maps defined on subsets of the set $\mathbb{R}$ of real numbers, which will be introduced in the next chapter.

**1.4.2. General properties of functions.** We say that a function $f : A \longrightarrow B$ is injective if for every element $b = f(A)$ is the image of only one $a \in A$. More formally:

**Definition 1.4.5**

A function $f : A \longrightarrow B$ is **injective** if

$$\forall a_1, a_2 \in A, \ \ a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$$

.

The following characterization is often useful:

*$f : A \longrightarrow B$ is injective if and only if*

$$\forall a_1, a_2 \in A \text{ such that } f(a_1) = f(a_2) \implies a_1 = a_2,$$

Another possible property, called surjectivety, consists in invading the whole codomain $B$ by the images of the elements of $A$:

**Definition 1.4.6**

A function $f : A \to B$ is **surjective** if

$$\forall b \in B, \ \ \exists a \text{ such that } f(a) = b,$$

or, in other words,

$$f(A) = B$$

Let us examine injectity and surgectivity of the above examples:

Example 1.4.7 (1)). *$A = \mathbb{N}$, $B = \{n + 100, \ \ n \in \mathbb{N}\}$ and $f : A \to B$ defined by setting $f(m) = m^2 + 200$, $\forall m \in \mathbb{N}$.*

*This map is injective. To see it let us use the characterization given after the definition. For any $m_1, m_2 \in \mathbb{N}$ one has*

$$f(m_1) = f(m_2) \Longleftrightarrow m_1^2 + 200 = m_2^2 + 200 \Longleftrightarrow m_1^2 = m_2^2 \Longleftrightarrow m_1 = m_2$$

*the last equivalence holding because both $m_1$ and $m_2$ are not negative.*

*Instead, $f$ is not surjective, because, for instance, $150 \in B$, but there are no $m \in A$ such that $f(m) = 150$. (Indeed, for any $m \in A$, $f(m) = m^2 + 200 \geq 200$.)*

Example 1.4.8. *Consider the domain $A = \mathbb{N} \times \mathbb{N}$ and the codomain $B = \{n - 100, \ \ n \in \mathbb{N}\}$ and the map $f : A \to B$ defined by setting $f(n_1, n_2) = n_1 n_2 - 50 \ \forall (n_1, n_2) \in \mathbb{N} \times \mathbb{N}$.*

*This function is not injective, because, for instance, $f(1, 2) = f(2, 1)$. Actually $f(n_1, n_2) = f(n_2, n_1)$, for all $(n_1, n_2) \in A$.*

*$f$ is not surjective: indeed, we have seen that*

$$f(A) = f(\mathbb{N} \times \mathbb{N}) = \{m - 50, \ \ m \in \mathbb{N}\} \subset \{n - 100, \ \ n \in \mathbb{N}\} = B,$$

*the last inclusion being strict. Notice that if we replaced B with $\tilde{B} = f(A) = \{m - 50, \ \ m \in \mathbb{N}\}$ we would recover surjectivity.* [11]

Example 1.4.9. *The function set $f : \mathbb{R} \to [0, 2]$ defined as $f(x) = (\sin x)^2$ is not injective. Indeed, for instance, one has $f(0) = f(2\pi) = 0$. Moreover, $f$ is not surjective, for*

$$f(\mathbb{R}) = f([\pi/2, \pi/2]) = f([0, \pi/2]) = [0, 1] \subset [0, 2] = B.$$

*Once again, if with replace the codomain $[0, 2]$ with $[0, 1]$, the map becomes surjective*

In the next chapters we will study many other examples in the special case when the domain and the codomain are subsets of $\mathbb{R}$.

[11] Of course this is a general fact, valid for every function $f : A \to B$: if one replaces the codomain $B$ with the new codomain $\tilde{B} = f(A)$ the function becomes surjective.

**Definition 1.4.10**

A map $f : A \to B$ which is both injective and surjective is said to be **bijective**.

In other words, a map $f : A \to B$ is bijective if and only if for every $b \in B$ there exists one and only one $a \in A$ such that $f(a) = b$. When a function $f : A \to B$ is bijective one also says that $f$ *establishes a one to one correspondence between A and B*.

Example 1.4.11. *There is a one to one correspondence between the natural numbers $\mathbb{N}$ and the set of odd numbers $\mathbb{O} := \{2n+1, \ \ n \in \mathbb{N}\}$. Indeed, one can verify that the function $f : \mathbb{N} \to \mathbb{O}$, $f(n) = 2n+1$, which is easily seen to be injective (indeed $f(n_1) = 2n_1 + 1 = 2n_2 + 1 = f(n_2) \implies 2n_1 = 2n_2 \implies n_1 = n_2$) is also surjective (if $m$ is odd, then $n = \frac{m-1}{2} \in \mathbb{N}$ and $f(n) = m$). Hence $f$ is bijective.*

Similarly there is a one-to-one correspondence between $\mathbb{N}$ and the set of even numbers.

We will see several other examples as soon as we will introduce the set of real numbers. Let us anticipate one of these examples:

Example 1.4.12. The map $f : [0, 2\pi] \to [-1, 1]$, $f(x) = \sin x \ \ \forall x \in [0, 2\pi]$ (see next Chapter) is not bijective, because it is not injective. However,the *restriction of* $f$ to the subdomain $[\pi/2, 3\pi/2]$, namely the map $f_{|_{[\pi/2,3\pi/2]}} : [\pi/2, 3\pi/2] \to [-1, 1]$, $f_{|_{[\pi/2,3\pi/2]}}(x) = f(x) \ \ \forall x \in [\pi/2, 3\pi/2]$, is bijective.

Example 1.4.13. The reader is invited to prove the following fact: *if $A$ is a set of $n$ elements and $B$ is a set of $m$ elements* $(n, m > 0)$*, then there exists (at least) one bijective map between A and B if and only if $n = m$.*

**1.4.3. Inverse functions.** The idea of inverse function of a (bijective) function $f$ consists in exchanging domain and codomain and mapping every element $y$ of the codomain into the element $x$ of the domain such that $y$ is th e$f$- image of $x$. Namely,

**Definition 1.4.14**

Let $f : A \to B$ be a bijective function. The *inverse* of $f$ is the function

$$f^{-1} : B \to A \qquad \forall y \in B, \quad f^{-1}(y) = x \in A \quad \text{provided } \ f(x) = y$$

Notice that the bijectivity of $f$ is essential. More precisely, the surjectivity guaranties that for every $y$ such an $x$ exists, while injectivity implies that the inverse is unique. (A posteriori, such an uniqueness justifies the fact that we say “the” inverse and not merely “an “ inverse).

Let us give some simple examples.[12]

Example 1.4.15. Consider the function $f : \mathbb{N} \to \mathbb{Q}$, $\quad f(n) = 6n + 17$. It is easy to prove that this function is injective: for any two natural numbers $n_1, n_2$, if $6n_1 + 17 = 6n_2 + 17$, then $6n_1 = 6n_2$ so that $n_1 = n_2$. Obviously this function is not surjective! For example there are not $n$ such that $f(n) = 1$ [13]

[12]Much more interesting and important examples will be given in the next chapter, as soo we will introduce elementary functions between subsets of rthe *real numbers*.

[13]By contradiction: if $1 = 6n + 17$,then $n = -\frac{8}{3}$, but $-\frac{8}{3}$ is not a natural number.

Of corse, by reducing the codomain of a function $f : A \to B$ to the image of the domain, one always recover surjectivity by considering the new function $\tilde{f} : A \to f(A)$ with $\tilde{f}(x) = f(x) \quad \forall x \in A$. So, in the previous case we can consider the function

$$\tilde{f} : \mathbb{N} \to C := \{6n + 17, \quad n \in \mathbb{N}\} \qquad \tilde{f}(n) := f(n) = 6n + 17 \quad \forall n \in \mathbb{N}$$

Now, $\tilde{f}$ is (injective and) surjective, hence it is bijective, so that there exists the inverse function. Actually one has

$$\tilde{f}^{-1} : C \to \mathbb{N}, \qquad f^{-1}(q) = \frac{q - 17}{6}$$

### 1.5. On counting subsets and their arrangements

In this section we answer to the following basic questions:

(1) Given a set with $k \geq 1$ elements, how many arrangements of these elements, i.e., *ordered* $k$-tuple can we construct? (By saying *ordered* we mean that, for instance, while $\{x, y, z\} = \{x, y, z\}$)
(2) Given a set with $n \geq 1$ elements, how many are its subsets (including the empty set)
(3) Given a set with $n \geq 1$ elements and $k \in \mathbb{N}$, $k \leq n$, how many subsets with $k$ elements do exist?

We will see that the answers are $k!$, $2^n$, and $\binom{n}{k} := \frac{n!}{k!(n-k)!}$, respectively where, for any natural number $m$, $m!$ denotes the *factorial of* $m$, namely

$$m! := m \cdot (m-1) \cdot (m-2) \cdot \ldots \cdot 2 \cdot 1$$

.

**Proposition 1.5.1**

For any natural number $k > 0$, the number of arrangenents of $k$ elements is $k!$

For instance, there are $9! = 732240$ different arrangements of a set of 9 elements.

Proof. Let us proceed by induction. The statement for $k = 1$ is obviously true. Now assume that it is true for $k$ and let us prove it for $k + 1$. So, from a set $S$ of $k + 1$ elements let us get out an element $\bar{s} \in S$. The set $S \setminus \{\bar{s}\}$ has $k$ elements, and by inductive hypothesis that there are $k!$ arrangements of them. How many arrangements of the whole set $S$ do we obtain? For any arrangement of $S \setminus \{\bar{s}\}$ we can insert $\bar{s}$ in some position, from the left of the first element until the right of the last element: this makes $n + 1$ possibilities. Therefore, the arrangements of $S$ are $k!(k + 1) = k + 1!$. □

By convention, one sets

$$0! = 1.$$

**Proposition 1.5.2**

A set with $n \geq 1$ elements has $2^n$ subsets.

Proof. We proceed by induction. If a set has 1 element, clearly it has $2 = 2^1$ subsets: itself and the empty set. Now suppose that the statement is true for some $n$: we wish to prove that it is true for $n + 1$

as well. Actually, if a set $S$ has $n+1$ elements and we choose one of them, say $\bar{s} \in S$, the subsets of $S$ split in two families: the family $\mathcal{C}$ of the subsets that contain $\bar{s}$ and the family $\mathcal{N}$ of the subsets that do not contain $\bar{s}$. But $\mathcal{N}$ coincides with the family of all subsets of $S\backslash\bar{s}$, and, by our inductive hypothesis, there are exactly $2^n$ of such subsets. On the other hand, each subset of the family $\mathcal{C}$ can be obtained as $E \cup \{\bar{s}\}$, where $E$ is a member of the family $\mathcal{N}$. Therefore the subsets in the family $\mathcal{C}$ are as many as the subsets in $\mathcal{N}$. Hence the number of *all* subsets of $S$ is equal to $2^n + 2^n = 2^{n+1}$, so the second inductive step is concluded, and the proposition is proved.

**Proposition 1.5.3**

Fix two natural numbers $k \leq n$. Let $\left(\begin{array}{c} n \\ k \end{array}\right)$ denote the number of subsets with $k$ elements. Then

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

Proof. If $k = 1$, the equality is verified, for there are exactly $n = \left(\begin{array}{c} n \\ 1 \end{array}\right)$ subsets with 1 element. If $k > 1$, let us begin with choosing the first element for a set of $k$ elements. There are $n$ possible choices. Secondly let us choose another element from the $n-1$ remaining ones. So far we have had $n(n-1)$. There remain $(n-2)$elements, so that at the third step we have $n-2$ choices. So far we have I had $n(n-1)(n-2)$ choices. Continuing similarly, our last choice among $n-k+1$ elements, having had a total of $n(n-1)\cdots(n-k+1)$ different possibilities in creating a subset of $k$ elements. However, in the above procedure different choices could end up with the same subset of $k$ elements which are only ordered differently.(For instance, if $S = \{1, 2, 3, 4\}$) and $k = 3$, with the above procedure we produce both $\{1, 3, 4\}$ and $\{4, 1, 3\}$, the former subset being equal to latter, only differently arranged). Therefore, we have to divide the obtained number $n(n-1)\cdots(n-k+1)$ by $k!$, which is the number of rearrangements of a given set of $k$ elements. Summing up, we have obtained

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n(n-1)\cdots(n-k+1)}{k!},$$

aso the proposition is proved. □

Let us observe that the binomial coefficient can also equivalently defined by

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{k!(n-k)!}$$

One also sets

$$\left(\begin{array}{c} n \\ 0 \end{array}\right) := 1,$$

which might interpreted as ”the number of subsets with 0 elements”: there is only one of such subsets, the empty set.

Since a set of $n$ elements has $2^n$ subsets, one immediately gets

**Corollary 1.5.4**

For every $1 \leq n$ one has
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

The following fact, which has an easy combinatorial meaning, will explain the construction rule of the Pascal triangle (see below).

**Proposition 1.5.5**

For every $1 \leq k < n$ one has
$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

Proof. Indded,
$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} = \frac{(n-1)!k}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!}$$
$$= \frac{(n-1)!n}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

**1.5.1. Newton's formula.** One refers to "Newton's formula" to mean the following elementary result:

**Proposition 1.5.6**

For every $a, b \in \mathbb{R}$ and every natural number $n \geq 1$, one has
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof. Writing
$$(a+b)^n = \underbrace{(a+b)}_{\text{factor } 1} \cdots \underbrace{(a+b)}_{\text{factor } n}$$
we understand that:

i) the result has to be a linear combination of terms of the form $a^{n-k}b^k$, $k = 0, \ldots, n$;

ii) if we consider the set
$$S = \{\text{factor } 1, ..., \text{factor } n\}$$

of the $n$ factors, for every $k \leq n$ the coefficient of the term $a^{n-k}b^k$ coincides with the number of subsets of $S$ made by $k$ elements. Since, as seen above, this number coincides with $\begin{pmatrix} n \\ k \end{pmatrix}$, the proof is concluded. □

Observe that, putting the last propositions together, we obtain the rule to build the well-known Pascal's triangle.

CHAPTER 2

# Real Numbers

## 2.1. Do we really need real numbers?

Numbers were invented to provide a quantitative basis to a large number of everyday life problems. The first set of numbers we learn in our life is the set of *naturals*, denoted by $\mathbb{N}$ and made by numbers $0, 1, 2, 3, \ldots$. This set naturally arises when we start with *count* objects. $\mathbb{N}$ is not simply a set of elements, because certain *operations* are defined: *sum*, *product* are well defined. They are *binary* operations in the sense that in each case we have two numbers and we compute a third one ($n+m$ or $n \cdot m$). In certain cases also *difference* and *division* are defined, despite not always (so we can compute $3-2$ or $6 \div 3$ while we cannot do $2-3$ or $3 \div 6$). An *order* is also defined, so we can always say if $n > m$ or $n = m$ or $n < m$ (we say that this order is *total*, in the sense that any two numbers can always be compared).[1]

Naturals are enough for a large number of common life situations, but sometimes we need to enlarge our abilities. For example: a firm earns $p$ from sales and has to pay $c$ to sustain her activity; it would be natural to compute $p-c$ to determine the profit or loss. If $c > p$ we would not be able to compute $p-c$ within naturals. That's why we need *integers* $\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \ldots\}$. On $\mathbb{Z}$ the same operations of $\mathbb{N}$ are defined but now difference can be computed between any two integers, thus $n-m$ makes sense for every $n, m \in \mathbb{Z}$). Yet, in general, we cannot always compute $n \div m$ when $n, m \in \mathbb{Z}$. Thus, if we have to divide a meter into three equal parts, that is, compute $1 \div 3$ we cannot do this within $\mathbb{Z}$. That is why we need a further enlargement, that is, *rational numbers* $\mathbb{Q} = \{\frac{m}{n} \ :\ n, m \in \mathbb{Z},\ n \neq 0\}$, which is also called also the "set of fractions". In $\mathbb{Q}$ we can do sum, difference, product, and division (by a number different from 0). As is well known, when the denominator of each expression is different from 0, one has

$$\frac{m}{n} \pm \frac{p}{q} = \frac{mq \pm np}{nq}, \quad \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}, \quad \frac{m}{n} \div \frac{p}{q} = \frac{mq}{np}.$$

Remarkably, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$, because we may identify numbers as $\frac{m}{1}$ with $m$. On $\mathbb{Q}$, a total order is also well defined.

It was well known to Greeks that rational numbers cannot exhaust all possible numbers. Take a square of side 1. According to Pythagoras' theorem, the diagonal of the square measures $\sqrt{1^2+1^2} = \sqrt{2}$.

**Theorem 2.1.1**

$\sqrt{2} \notin \mathbb{Q}$.

[1]See below for a rigorous definition of total order

PROOF. Assume, by contradiction, $\sqrt{2} \in \mathbb{Q}$, that is $\sqrt{2} = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$. We may assume that $m, n > 0$ (thus $m, n \in \mathbb{N}$) and they do not share any common divisor. **Our goal is to prove that, necessarily, they must have a common divisor**, contradicting the previous assumption. To show this, notice that

$$\frac{m^2}{n^2} = 2, \iff m^2 = 2n^2, \iff m^2 \text{ is even.}$$

*What about $m$?* Guess: *$m$ itself must be even*. Indeed, if $m$ were odd, say $m = 2k + 1$, then

$$m^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2) + 1 \implies m^2 \text{ is odd.}$$

Thus, $m = 2k$ (that is 2 divides $m$). Hence,

$$m^2 = 2n^2, \iff 4k^2 = 2n^2, \iff n^2 = 2k^2, \iff n^2 \text{ is even.}$$

By the same argument, $n$ itself is even, that is 2 divides also $n$. We obtained that 2 divides both $m, n$ and this contradicts our initial assumption. □

### 2.2. Axiomatic definition of $\mathbb{R}$

We just learnt that there are numbers that are not rational numbers, called *irrational numbers*. The construction of the extension of $\mathbb{Q}$ is technically difficult and beyond the scope of this course, hence we will omit here. We will just buy this as a

**Theorem 2.2.1: Existence of $\mathbb{R}$**

There exists a set, denoted by $\mathbb{R} \supset \mathbb{Q}$, called **set of real numbers** fulfilling the following properties.

- On $\mathbb{R}$ a **sum** $+$ is defined, coincident with the ordinary sum on $\mathbb{Q}$, such that
  i) (**associativity**): $x + (y + z) = (x + y) + z$, $\forall x, y, z \in \mathbb{R}$.
  ii) (**commutativity**): $x + y = y + x$, $\forall x, y \in \mathbb{R}$.
  iii) (**zero**): $x + 0 = x$, $\forall x \in \mathbb{R}$.
  iv) (**opposite**): $\forall x \in \mathbb{R}$, $\exists! y \in \mathbb{R} : x + y = 0$. The opposite is denoted by $-x$.
- On $\mathbb{R}$ a **product** $\cdot$ is defined, coincident with the ordinary product on $\mathbb{Q}$, such that
  i) (**associativity**): $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $\forall x, y, z \in \mathbb{R}$.
  ii) (**commutativity**): $x \cdot y = y \cdot x$, $\forall x, y \in \mathbb{R}$.
  iii) (**unit**): $x \cdot 1 = x$, $\forall x \in \mathbb{R}$.
  iv) (**reciprocal**): $\forall x \in \mathbb{R}\backslash\{0\}$, $\exists! y \in \mathbb{R} : x \cdot y = 1$. The reciprocal is denoted by $\frac{1}{x}$.
- Sum and product fulfil **distributivity**,

$$x \cdot (y + z) = x \cdot y + x \cdot, \forall x, y, z \in \mathbb{R}.$$

- On $\mathbb{R}$ it is also defined an **ordering** $<$, coinciding with the order on $\mathbb{Q}$, such that:
  i) (**total ordering**): $\forall x, y \in \mathbb{R}$ only one between $x < y$, $x = y$, $y < x$ is true.
  We write $x \leqslant y$ if $x < y$ or $x = y$. Then
  ii) (**transitivity**): if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$.
  iii) (**reflexivity**): if $x \leqslant y$ and $y \leqslant x$ then $y = x$.
  iv) (**invariance by sum**): if $x \leqslant y$ then $x + z \leqslant y + z$, $\forall z \in \mathbb{R}$.
  v) (**invariance by product**): if $x \leqslant y$ then $x \cdot z \leqslant y \cdot z$, $\forall z \in \mathbb{R},\ z \geqslant 0$.

Finally, on $\mathbb{R}$ the following property holds true:

- (**completeness**): any **lower bounded** (or **upper bounded**) set $S \subset \mathbb{R}$ admits **best lower bound** (**best upper bound**).

Completeness is the trademark of $\mathbb{R}$. To clarify this property, we need to fix some definitions:

**Definition 2.2.2: (lower bound)**

A set $S \subset \mathbb{R}$ is **lower bounded** if

$$\exists a \in \mathbb{R} \ : \ a \leqslant s, \ \forall s \in S.$$

**Definition 2.2.3: (upper bound)**

A set $S \subset \mathbb{R}$ is **upper bounded** if

$$\exists b \in \mathbb{R} \ : \ b \geqslant s, \ \forall s \in S.$$

If a set $S$ is both lower and upper bounded, we say that it is **bounded**. In particular:

$$S \textbf{ bounded} \iff \exists a, b \in \mathbb{R} \ : \ a \leqslant s \leqslant b, \ \forall s \in S.$$

We are now ready to define the crucial concept of *best lower bound* or, more commonly, *infimum* :

**Definition 2.2.4: (best lower bound)**

Given a lower bounded set $S \subset \mathbb{R}$, a number $\alpha \in \mathbb{R}$ is called **best lower bound** or, more commonly, **infimum** of $S$ (notation $\alpha = \inf S$) if

i) $\alpha$ is a lower bound for $S$, that is $\alpha \leqslant s$, $\forall s \in S$;

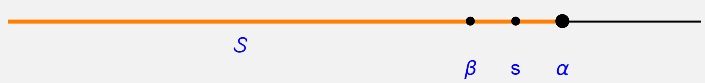
ii) $\alpha$ is the best lower bound for $S$, that is $\forall \beta > \alpha$, $\exists s \in S$ such that $\alpha \leqslant s \leqslant \beta$.


**Definition 2.2.5: (best upper bound)**

Given a lower bounded set $S \subset \mathbb{R}$, a number $\gamma \in \mathbb{R}$ is called **best upper bound** or, more commonly, **supremum** of $S$ (notation $\gamma = \sup S$) if

i) $\gamma$ is an upper bound for $S$, that is $\gamma \geqslant s$, $\forall s \in S$;

ii) $\gamma$ is the best upper bound for $S$, that is $\forall \beta < \gamma$, $\exists s \in S$ such that $\gamma \leqslant s \leqslant \beta$.




The infimum and the supremum are weakened versions of two extremely important concepts, *minimum* and *maximum* of a set.

**Definition 2.2.6**

Given $S \subset \mathbb{R}$ we say that $m \in \mathbb{R}$ is the **minimum of** $S$ (notation $m = \min S$) if

i) $m \in S$

ii) $m \leqslant s$, $\forall s \in S$

**Definition 2.2.7**

Given $S \subset \mathbb{R}$ we say that $M \in \mathbb{R}$ is the **maximum of** $S$ (notation $M = \max S$) if

i) $M \in S$

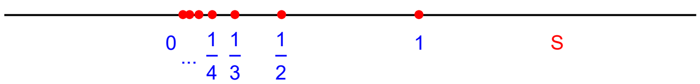
ii) $M \geqslant s$, $\forall s \in S$

If the set $S$ is **finite**, that is made by a finite number of elements, it is clear that both $\min S$ and $\max S$ exist and they can be determined by a simple algorithm. When $S$ is **infinite**, that is not made by a finite number of elements, the existence of min/max is no more sure and, in general, it is false as the following example shows:

Example 2.2.8. *The set*

$$S := \left\{\frac{1}{n} \;:\; n \in \mathbb{N},\; n \neq 0\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\},$$

*has a maximum but it has not a minimum.*




Sol. — Clearly $0 < \frac{1}{n} \leqslant 1$ for every $n \in \mathbb{N}\backslash\{0\}$, and $1 \in S$: previous inequality implies that $1 = \max S$. As for $\min S$, the argument is more delicate. We may guess $\min S = 0$. However, $0 \notin S$ (indeed it is not possible $\frac{1}{n} = 0$), thus 0 cannot be a minimum for $S$ (remember: minimum and maximum must be elements of $S$!). This, however, does not disprove that $S$ might have a minimum, it proves only that 0 cannot be the minimum. Therefore, what if $\min S = p$ for some other $p$? If this were the case, then $p \in S$ so $p = \frac{1}{N}$ for some $N \in \mathbb{N}\backslash\{0\}$. However, then, since

$$S \ni \frac{1}{N+1} < \frac{1}{N}.$$

We would have an element of $S$ strictly lower than the supposed $\min S$ and this is a contradiction! Therefore, we have to conclude that $\min S$ does not exists. □

As we anticipated, $\inf S$ is a weakened version of $\min S$. The sense of this statement is made precise by the following

**Proposition 2.2.9**

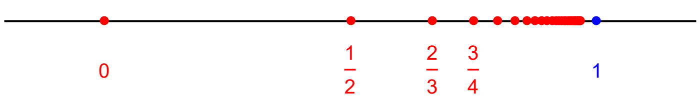
- If $\exists \min S$, then $\min S = \inf S$
- if $\exists \max S$, then $\max S = \sup S$.

The proof is left as an exercise (simple and recommended).

Example 2.2.10. *Let*

$$S = \left\{\frac{n-1}{n} \;:\; n \in \mathbb{N},\; n > 0\right\}.$$

*Discuss* $\inf S$*,* $\sup S$ *and existence of* $\min S$*,* $\max S$*.*




Sol. — Clearly $0 \leqslant \frac{n-1}{n} < 1$ for every $n \in \mathbb{N}$, $n > 0$ (that is $n \geqslant 1$). In particular, $S$ is bounded thus it admits both $\inf S$ and $\sup S$. Guess:

$$\min S = 0, \text{ (thus } \inf S = 0), \quad \sup S = 1, \text{ and it does not exist } \max S$$

Let us prove our guess: first $\min S = 0$. Indeed

- $0 \in S$ (take $n = 1$: $\frac{n-1}{n} = 0$);
- $0 \leqslant s$, $\forall s \in S$: indeed this means $0 \leqslant \frac{n-1}{n}$, $\forall n \in \mathbb{N}$, $n > 0$ (already noticed).

Thus, in particular, $\inf S = 0$ (general fact). Second claim: $\sup S = 1$. Indeed

- $1 \geqslant \frac{n-1}{n}$, $\forall n \in \mathbb{N}$, $n > 0$ (already noticed);
- 1 is the best upper bound: we have to check that
$$\forall \beta < 1,\ \exists n \in \mathbb{N},\ :\ \frac{n-1}{n} > \beta.$$
However,
$$\frac{n-1}{n} = 1 - \frac{1}{n} > \beta,\ \Longleftrightarrow\ \frac{1}{n} < 1 - \beta,\ \overset{\beta<1}{\Longleftrightarrow}\ n > \frac{1}{1-\beta}.$$
Thus, taking $n > \frac{1}{1-\beta}$ we have the conclusion.

Finally: $\max S$ does not exist. Indeed, if $\max S$ would exists, then necessarily $\max S = \sup S = 1$. However, $1 \notin S$ (because $1 - \frac{1}{n} = 1$ if and only if $\frac{1}{n} = 0$, impossible). □

In the final part of the example, we find an element of $s > \beta$ provided $n$ is greater than a suitable real number. This seems to be obvious. However, it is not immediate to find a proper justification. Indeed, the question is: *are there real numbers greater than any integer?* This is a nontrivial remarkable property:

**Theorem 2.2.11: Archimedean property**

$$\forall b \in \mathbb{R},\ \exists n \in \mathbb{N},\ :\ n > b.$$

Proof. Suppose that the conclusion is false and that there exists a real $b$ such that
$$n \leqslant b,\ \forall n \in \mathbb{N}.$$
$\mathbb{N}$ would be then upper bounded in $\mathbb{R}$, so by *completeness* $\sup S$ exists
$$\mathbb{R} \ni \alpha := \sup \mathbb{N}$$
Let us give a look at this weird situation with the help of a figure:


Now, taking any $\alpha - 1 < \beta < \alpha$, since $\alpha$ is the best upper bound,
$$\exists \widetilde{n} \in \mathbb{N},\ :\ \beta < \widetilde{n} \leq \alpha.$$
But then $\mathbb{N} \ni \widetilde{n} + 1 > \beta + 1 = \alpha - 1 + 1 = \alpha$: so we have found an element of $\mathbb{N}$, namely $\widetilde{n} + 1$, strictly bigger than $\alpha = \sup \mathbb{N}$: this is a contradiction! □

Example 2.2.12. *Find* $\inf/\sup$ *and, if any,* $\min/\max$ *of*
$$S := \left\{ 2 - \frac{1}{\sqrt{n}}\ :\ n \in \mathbb{N},\ n \geqslant 1 \right\}.$$

Sol. — $\sup/\max$. The elements of $S$ are numbers
$$2 - \frac{1}{\sqrt{n}},$$
as $n \in \mathbb{N}$, $n \geqslant 1$. Clearly, $2 - \frac{1}{\sqrt{n}} < 2$ for any $n \geqslant 1$. We deduce that $S$ is upper bounded (by 2 for instance), therefore $\sup S \in \mathbb{R}$ by completeness. Let us see that $\sup S = 2$. Intuitively, it is clear: as $n$ is "big", the number

$2 - \frac{1}{\sqrt{n}}$ is close to 2 and lower than it. We have already seen that 2 is an upper bound, it remains to check that it is the best, that is

$$\forall \beta < 2,\ \exists n \geqslant 1\ :\ 2 - \frac{1}{\sqrt{n}} \geqslant \beta.$$

But

$$2 - \frac{1}{\sqrt{n}} \geqslant \beta,\ \iff\ \frac{1}{\sqrt{n}} \leqslant 2 - \beta,\ \iff\ \sqrt{n} \geqslant \frac{1}{2-\beta},\ \iff\ n \geqslant \left(\frac{1}{2-\beta}\right)^2.$$

Therefore, if $n \geqslant \left(\frac{1}{2-\beta}\right)^2$ (precisely $n \geqslant \left[\left(\frac{1}{2-\beta}\right)^2\right] + 1$) we are done. About max: if it exists it coincides with $\sup S = 2$. Thus, we have to check if $2 \in S$ or not. However, $2 - \frac{1}{\sqrt{n}} < 2$ for any $n \geqslant 1$, so $2 \notin S$, therefore $\max S$ does not exist.

Let us look for the infimum or the minimum. . Notice that if $n \nearrow$ (that is $n$ increase) then $2 - \frac{1}{\sqrt{n}} \nearrow$ (that is $2 - \frac{1}{\sqrt{n}}$ increase ). Therefore

$$S \ni 2 - \frac{1}{\sqrt{1}} = 1 \leqslant 2 - \frac{1}{\sqrt{n}},\ \forall n \geqslant 1,\quad \Longrightarrow\quad 1 = \min S = \inf S. \quad \square$$

Of course, a set $S \subset \mathbb{R}$ might be not lower/upper bounded. In these cases, we say that

- $S$ is **lower unbounded** if $\forall \beta \in \mathbb{R},\ \exists s \in S,\ :\ s \leqslant \beta$.
- $S$ is **upper unbounded** if $\forall \beta \in \mathbb{R},\ \exists s \in S,\ :\ s \geqslant \beta$.

For convenience, we will write

$$\inf S = -\infty,\ \text{when } S \text{ is lower unbounded},\quad \sup S = +\infty,\ \text{when } S \text{ is upper unbounded}.$$

But, warning: $\pm\infty$ **are not numbers**! We introduce some useful notations. Let $a, b \in \mathbb{R}$, $a < b$. We set

$$[a,b] := \{x \in \mathbb{R}\ :\ a \leqslant x \leqslant b\}. \quad ]a,b] := \{x \in \mathbb{R}\ :\ a < x \leqslant b\}.$$

$$[a,b[ := \{x \in \mathbb{R}\ :\ a \leqslant x < b\}. \quad ]a,b[ := \{x \in \mathbb{R}\ :\ a < x < b\}.$$

$$]-\infty,b] := \{x \in \mathbb{R}\ :\ x \leqslant b\}. \quad ]-\infty,b[ := \{x \in \mathbb{R}\ :\ x < b\}.$$

$$[a,+\infty[ := \{x \in \mathbb{R}\ :\ x \geqslant a\}. \quad ]a,+\infty[ := \{x \in \mathbb{R}\ :\ x > a\}.$$

All these sets are called **intervals**. We set also $]-\infty,+\infty[ := \mathbb{R}$.

Let $x \geqslant 0$. Sooner or later, some integer $n$ will be bigger than $x$, that is $x \in [0,n[$. Now

$$[0,n[ = [0,1[ \cup [1,2[ \cup [2,3[ \cup \ldots \cup [n-1,n[.$$

And because the intervals $[k,k+1[$ are disjoint, $x$ belongs only to exactly one of them.

**Definition 2.2.13**

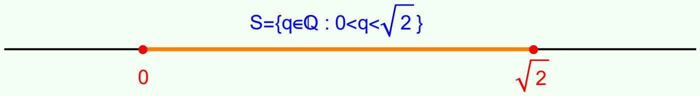
Given $x > 0$ we call **integer part of** $x$ the number $[x] \in \mathbb{N}$ such that
$$[x] \leqslant x < [x] + 1.$$
We call **fractional part of** $x$ the number $x - [x] =: \{x\}$.

Integer part is defined also for $x < 0$: $[x] \in \mathbb{Z}$ is the unique integer such that
$$[x] \leqslant x < [x] + 1.$$
In this case, for instance $[-\frac{3}{2}] = -2$.

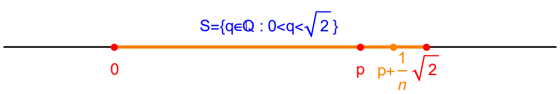
**2.2.1. Supplement: $\mathbb{Q}$ does not fulfil completeness.** Real numbers are really more effective than rational numbers. Apart from completeness, both $\mathbb{R}$ and $\mathbb{Q}$ have the same algebraic structure. What makes the difference between the two is precisely completeness, which is true in $\mathbb{R}$ while it is false in $\mathbb{Q}$.

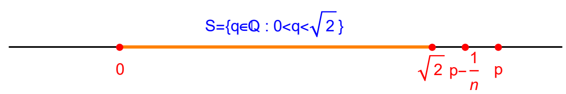
**Proposition 2.2.14**

The set $S := \{q \in \mathbb{Q} \ : \ 0 < q < \sqrt{2}\}$ is bounded but it does not admits $\sup S$ in $\mathbb{Q}$.




Proof. Clearly $S$ is bounded, thus it admits $p := \sup S$. We claim $p^2 = 2$ (that is, $p = \sqrt{2}$). If this holds, in particular $\sup S \notin \mathbb{Q}$, thus $S$ cannot have a best upper bound in $\mathbb{Q}$. This is the strategy, but the proof of our claim, namely $p = \sqrt{2}$, is not so easy.




*Let us prove that* $p = \sqrt{2}$. Assume first $p = \sup S > \sqrt{2}$. As the right figure suggests, $p$ shouldn't be the *best* upper bound, that is we may find $p - \frac{1}{n}$ with $n$ big enough in such a way that

$$\sqrt{2} < p - \frac{1}{n} < p. \tag{2.2.1}$$

Indeed, notice that $p - \frac{1}{n} < p$ is clearly true whatever is $n \in \mathbb{N}$, $n \neq 0$. As for the first inequality of (2.2.1), observe that
$$\sqrt{2} < p - \frac{1}{n}, \iff 2 < \left(p - \frac{1}{n}\right)^2 = p^2 - \frac{2}{n}p + \frac{1}{n^2}.$$
Clearly, if we choose $n$ such that
$$2 < p^2 - \frac{2}{n}p, \iff \frac{2}{n}p < p^2 - 2, \iff n > \frac{2p}{p^2 - 2}.$$
we get
$$p^2 - \frac{2}{n}p + \frac{1}{n^2} > p^2 - \frac{2}{n}p > \sqrt{2}$$

(2.2.1) is finally satisfied.

Assume now that $p = \sup S < \sqrt{2}$. As the left figure suggests, we should be able to find ad $n \in \mathbb{N}$ such that

$$p < p + \frac{1}{n} < \sqrt{2}. \tag{2.2.2}$$

If this is the case, since $p + \frac{1}{n} \in \mathbb{Q}$ we would have $p + \frac{1}{n} \in S$: but then $p$ would not be an upper bound for $S$ because there's some element of $S$ strictly greater than $p$. Now, to find such an $n$ we have to solve

$$p + \frac{1}{n} < \sqrt{2}, \iff \left(p + \frac{1}{n}\right)^2 < 2, \iff p^2 + \frac{2}{n}p + \frac{1}{n^2} < 2$$

Since $\frac{1}{n^2} < \frac{1}{n}$ for every $n \geq 1$, if we are able to solve

$$p^2 + \frac{2}{n}p + \frac{1}{n} < 2$$

we are done. Now

$$p^2 + \frac{2}{n}p + \frac{1}{n} < 2, \iff \frac{1}{n}(2p+1) < 2 - p^2, \iff n > \frac{2p+1}{2-p^2}.$$

Hence the second inequality of (2.2.2) is valid for such an $n$. **Conclusion:** we proved that both $p > \sqrt{2}$ and $p < \sqrt{2}$ lead to a contradiction. Thus, $p = \sqrt{2}$. □

### 2.3. "How many elements in a set?"

The notion of *finite set* is common intuition: more less, a set $E$ will be said *finite* if "one can count its elements". How do we translate this idea in rigorous mathematical language? A simple idea consists in considering the following sample sets:

$$S_1 := \{1\} \qquad S_2 := \{1,2\} \qquad S_3 := \{1,2,3\} \qquad \ldots, \qquad S_9 := \{1,2,3,\ldots 9\}$$

and, in general, for every $n \in \mathbb{N}$,

$$S_n := \{1,2,3,\ldots n\}$$

.

The definition of *finite set* is then as follows:

**Definition 2.3.1**

A set $E \neq \emptyset$ is **finite** if there exists a bijective function between $E$ and $S_n$, for some positive natural number $n$. Otherwise said, a set $E$ is finite if, for some positive natural number $n$, there is a one-to-one correspondence between $E$ and$S_n$. In this case one can say that $E$ *has n elements*.
A set $E \neq \emptyset$ is **infinite** if it is not finite.

Clearly, the set $E$ of even numbers, the set $O$ of odd numbers, and the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are infinite. We will see the infinite sets $E$, $O$, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ can be pairwise put into one-to-one relationship. Instead, this is not true for $\mathbb{R}$, which is *much bigger* than these sets, (in the sense of *cardinality*, see Def. below)

## 2.4. Elementary functions

*Elementary functions* are some particular functions on which many others are built. Despite the name, elementary functions are not at all elementary in the sense that in most cases we do not dispose of elementary formulas to compute their values. In this section, we will motivate their introduction and give precise formal definitions and properties. Proofs are much beyond our scope and are not necessary to work with these functions, so we will skip. Later, when we will discuss Differential Calculus, we will see how the problem of numerical computation can be solved.

Before constructing elementary functions let us give a few definitions on *order* and *monotonicity.*

**Definition 2.4.1**

An *order* on a set $K$ is a binary relation "$\preceq$" on $K$ verifying *riflexivity*, i.e. $k \preceq k \ \forall k \in K$,*antisimmetry*, i.e. $k_1 \preceq k_2$ and $k_2 \preceq k_1 \implies k_1 = k_2$ and *transitivity*, i.e. $k_1 \preceq k_2$ and $k_2 \preceq k_3 \implies k_1 \preceq k_3$. An order is *total* if , for any $k_1, k_2 \in K$ one has $k_1 \preceq k_2$ or $k_2 \preceq k_1$, (i.e all pairs of elements can be compared).

The order $\leq$ on $\mathbb{Q}$ is total. An example of not total order is given by the inclusion relation $\subseteq$ on the set $K = \mathcal{P}(\mathbb{R})$ whose elements are the subsets of $\mathbb{R}$. Namely, for all $A, B \in \mathcal{P}(\mathbb{R})$ one can define $A \preceq B$ as $A \subseteq B$ (i.e. $A$ is contained in $B$). This is an order, indeed: $A \subseteq A \implies A = A$ (reflexivity), $A \subseteq B$ and $B \subseteq A$ implies $A = B$ (antisimmetry), $A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$(transitivity). However, this order is not total. For instance, if $A\{\frac{2}{3}, 5, \pi, 25\}$ and $B\{\frac{7}{8}, 5, 10, 2022\}$, it is neither $A \subseteq B$ nor $B \subseteq A$.

**Definition 2.4.2**

Let $f : D \subseteq \mathbb{R} \longrightarrow E \subseteq \mathbb{R}$. We say that $f$ is **increasing** (notation $f \nearrow$) if

$$f(x) \leqslant f(y), \ \forall x, y \in D \ : \ x \leqslant y.$$

When $f(x) < f(y)$ as $x < y$ we say that $f$ is **strictly increasing**

For instance it is well known that the function $f = \sin : [0, \pi/2] \to \mathbb{R}$ as well as the *identity* map[2] $id : \mathbb{R} \to \mathbb{R}$, $id(x) := x$, are strictly increasing functions.

Similarly, one defines decreasing functions:

**Definition 2.4.3**

Let $f : D \subseteq \mathbb{R} \longrightarrow E \subseteq \mathbb{R}$. We say that $f$ is **decreasing** (notation $f \searrow$) if

$$f(x) \geqslant f(y), \ \forall x, y \in D \ : \ x \leqslant y.$$

When $f(x) > f(y)$ as $x < y$ we say that $f$ is **strictly decreasing**.

An increasing/decreasing function is said generically **monotone**.

It is almost immediate to verify that strict monotonicity guarantees injectivity:

[2]The word *map* is a synonymous of the word *function*

**Theorem 2.4.4**

If a function $f : D \subseteq \mathbb{R} \longrightarrow E \subseteq \mathbb{R}$ is is strictly increasing or strictly decreasing then it is injective.

Indeed, if, for instance, $f$ is strictly decreasing and $x_1, x_2 \in E$, $x_1 \neq x_2$, up to renaming $x_1, x_2$ we can assume that $x_1 < x_2$. Then, by monotonicity, one gets $f(x_2) < f(x_1)$, so that $f(x_2) \neq f(x_1)$, so injectivity is proved. An analogous argument works for the case when $f$ is strictly increasing.

**2.4.1. Powers.** Powers are functions of type $x^\alpha$ where $\alpha$ is some exponent. There are several different types of powers according to the type of exponent. The simplest case is $x^n$ with $n \in \mathbb{N}\ n \geqslant 1$: $x, x^2, x^3, \ldots$. These functions are defined for every $x \in \mathbb{R}$. Moreover

$$(-x)^n = x^n,\ \forall x \in \mathbb{R},\ n \text{ even}, \quad (-x)^n = -x^n,\ \forall x \in \mathbb{R},\ n \text{ odd}.$$

Powers fulfil simple elementary properties:

i) $x^n x^m = x^{n+m}$;
ii) $(x^n)^m = x^{nm}$;
iii) if $0 \leqslant x < y$ then $x^n < y^n$ for every $n \geqslant 1$.

Next case is $x^n$, with $n \in \mathbb{Z}$, $n < 0$. The natural definition is

$$x^n := \frac{1}{x^{-n}}.$$

This power is well defined for every $x \neq 0$ and fulfils i) and ii) while iii) is simply reversed, that is $0 < x < y$ implies $x^n > y^n$ for $n \leqslant -1$.

The first nontrivial case is $x^q$ with $q = \frac{m}{n} \in \mathbb{Q}$. Since we wish ii) be verified, we may expect

$$x^{\frac{m}{n}} = \left(x^{1/n}\right)^m.$$

Thus, we are reduced to define $x^{1/n}$. Since $n = 0$ doesn't make sense, $n = 1$ is trivial, we start by the case $n \in \mathbb{N}$, $n \geqslant 2$. Again, since by ii) we expect

$$\left(x^{\frac{1}{n}}\right)^n = x^1 = x,$$

$x^{1/n}$ would be what we call the *$n-$th root of the number $x$* also denoted by $\sqrt[n]{x}$. As we have seen with $\sqrt{2}$, the root might not be defined in $\mathbb{Q}$. However, thanks to completeness, it turns out that the $n-$th root of a positive number always exists in $\mathbb{R}$: this is the content of the following nontrivial

**Theorem 2.4.5**

Fiy $n \in \mathbb{N}$, $n \geqslant 2$. Then

$$\forall y \geqslant 0,\ \exists!\, x \geqslant 0,\ :\ x^n = y.$$

We call such unique $x$ the **$n-$th root of** $y$ and we denote it by $\sqrt[n]{y}$. It turns out that

$$\sqrt[n]{y} = \sup\{x > 0\ :\ x^n < y\}. \tag{2.4.1}$$

Using the notion of *inverse function* we can rephrase this theorem as follows

**Theorem 2.4.6**

Fix $n \in \mathbb{N}$, $n \geqslant 2$. Then the function $f : [0,+\infty[ \to [0,+\infty[$, $f(x) := x^n$ is bijective.[a] Its inverse function $f^{-1} : [0,+\infty[ \to [0,+\infty[$, usually denoted by $\sqrt[n]{\ }$, is called the $n-$**th root** and is defined by (2.4.1).

[a]It is injective because it is strictly increasing and it is surjective by (2.4.1) and the completeness of $\mathbb{R}$.

Clearly, for practical purposes, (2.4.1) is of no use: we cannot expect to compute $\sqrt{2}$ by computing $\sup\{y > 0 \ : \ y^2 < 2\}$. This is, for the moment, a strong limitation. As we anticipated, Differential Calculus will provide a method for numerical computation of roots.

*If n is odd* we can define the $n-$th root also for negative numbers posing

$$\sqrt[n]{x} := -\sqrt[n]{-x}, \text{ if } x < 0, \ (n \text{ odd}). \tag{2.4.2}$$

Indeed, this definition works as the $n-$th root of $x < 0$ because

$$\left(\sqrt[n]{x}\right)^n \overset{(2.4.2)}{=} \left(-\sqrt[n]{-x}\right)^n \overset{n\ odd}{=} -\left(\sqrt[n]{-x}\right)^n = -(-x) = x.$$

Therefore

$$x^{1/n}, \text{ is defined for } \begin{cases} x \in [0,+\infty[, \ n = 2,4,6,\dots, \\ \\ x \in ]-\infty,+\infty[, \ n = 1,3,5,\dots. \end{cases}$$

To pass to $x^q$ with $q \in \mathbb{Q}$ is now easy: we set

$$x^{\frac{m}{n}} := \left(x^{\frac{1}{n}}\right)^m.$$

There is just a little care to have here: a rational number $q = \frac{m}{n}$ can be written in infinitely many ways. We will agree to consider the fraction $\frac{m}{n}$ as reduced as much as possible, that is $m$ and $n$ without common divisors; moreover we can always assume $n > 0$. With this agreement, if $m > 0$,

$$x^{\frac{m}{n}} := \left(x^{\frac{1}{n}}\right)^m, \quad \begin{cases} \forall x \in [0,+\infty[, \ (n, \ \text{even}), \\ \\ \forall x \in ]-\infty,+\infty[, \ (n, \ \text{odd}) \end{cases}$$

When $m < 0$ the definition is the same except that in this case the power is not defined for $x = 0$. Power $x^q$, $q \in \mathbb{Q}$, fulfils properties at all similar to those for $x^n$:

- $x^{p+q} = x^p x^q$;
- $(x^p)^q = x^{pq}$;
- $0^p = 0$, for any $p > 0$;
- $1^p = 1$ for any $q \in \mathbb{Q}$.

Moreover, we have the following **monotonicity properties**:

$$\text{if } 0 < x < y, \text{ then } \begin{cases} x^p < y^p, & p > 0; \\ \\ x^p > y^p, & p < 0. \end{cases} \qquad \text{if } p < q, \text{ then } \begin{cases} x^p < x^q, & x > 1; \\ \\ x^p > x^q, & x < 1. \end{cases}$$

As functions, rational powers $x^q$ are

$$q = \frac{m}{n} \in \mathbb{Q}\backslash\mathbb{Z}, \quad \begin{cases} x^q : [0, +\infty[ \longrightarrow [0, +\infty[, & q > 0, \ n \text{ even}, \\ x^q : ]0, +\infty[ \longrightarrow ]0, +\infty[, & q < 0, \ n \text{ even}, \\ x^q : ]-\infty, +\infty[ \longrightarrow ]-\infty, +\infty[, & q > 0, \ n \text{ odd}, \\ x^q : ]-\infty, +\infty[\backslash\{0\} \longrightarrow ]-\infty, +\infty[\backslash\{0\}, & q < 0, \ n \text{ odd}. \end{cases}$$

The last step is the definition of power $x^\alpha$ with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. We need this type of powers to define functions like *exponentials* and *logarithms*. However, it is not easy to understand what it should mean $x^{\sqrt{2}}$ for instance. To fix ideas, let $x > 1$ and notice that, by power monotonicity,

$$x^q < x^p, \ \forall q < p < \alpha.$$

In other words, we may expect that

$$x^\alpha := \sup\{x^q \ : \ q \in \mathbb{Q}, \ q < \alpha\}. \tag{2.4.3}$$

We can take (2.4.3) as a definition (completeness ensures well-posedness):

**Theorem 2.4.7**

Let $\alpha \in \mathbb{R}$ and $x > 0$. If $x > 1$ set

$$x^\alpha = \sup\{x^q \ : \ q \in \mathbb{Q}, \ q < \alpha\}.$$

If $x = 1$, $1^\alpha := 1$, while if $x < 1$ we set

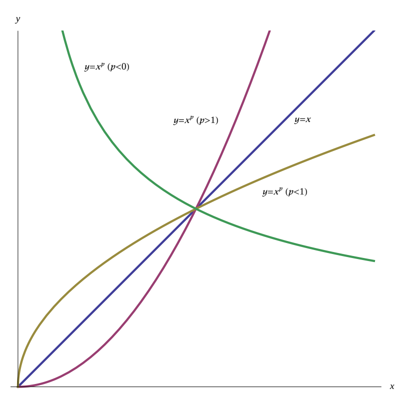
$$x^\alpha := \frac{1}{x^{-\alpha}}.$$

Then, the following properties hold:

- $x^{\alpha+\beta} = x^\alpha x^\beta$.
- $(x^\alpha)^\beta = x^{\alpha\beta}$, in particular $x^{-\alpha} = \frac{1}{x^\alpha}$.
- $x^\alpha \nearrow$ strictly if $\alpha > 0$, $x^\alpha \searrow$ strictly if $\alpha < 0$.

As function $x^\alpha$ is bijective between $I = ]0, +\infty[$ and $J = ]0, +\infty[$ (with $(x^\alpha)^{-1} = x^{1/\alpha}$).

Here below the graph of powers for different values of $\alpha$.



**2.4.2. Exponentials and Logarithms.** Fix a positive number $a > 0$ (called **base**) and, for every $x \in \mathbb{R}$ define

$$\exp_a(x) := a^x.$$

Since powers of $a > 0$ make sense for every $x \in \mathbb{R}$ exponent, we have a function

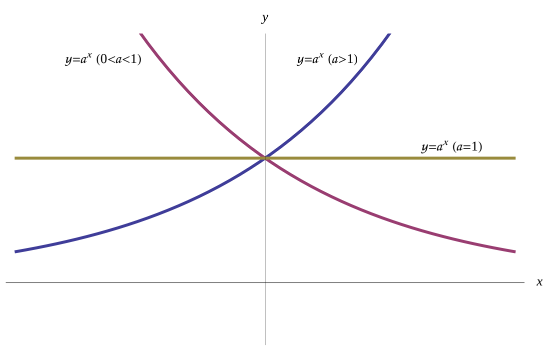
$$\exp_a : ]-\infty, +\infty[ \longrightarrow ]0, +\infty[, \quad \exp_a(x) := a^x, \ x \in \mathbb{R},$$

that we call **exponential of base** $a$. Clearly $\exp_1 \equiv 1$. From the properties of powers, we easily have

**Theorem 2.4.8**

Let $a > 0$, $a \neq 1$. Then

- $\exp_a \nearrow$ strictly if $a > 1$, $\exp_a \searrow$ strictly if $a < 1$.
- $a^{x+y} = a^x a^y$, $\forall x, y \in \mathbb{R}$.
- $\exp_a(0) = a^0 = 1$, $\forall a > 0$.




Example 2.4.9. *Solve*

$$5^x - \frac{1}{5^{x-1}} \geqslant 4.$$

Sol. — Being $5^{x-1} > 0$ the inequality makes sense for any $x \in \mathbb{R}$. We have,

$$5^x - \frac{1}{5^{x-1}} \geqslant 4, \iff 5^{2x-1} - 1 \geqslant 4 \cdot 5^{x-1}, \iff \frac{1}{5}5^{2x} - \frac{4}{5}5^x - 1 \geqslant 0, \iff 5^{2x} - 4 \cdot 5^x - 5 \geqslant 0.$$

Therefore, setting $y = 5^x$, we have

$$5^{2x} - 4 \cdot 5^x - 5 \geqslant 0, \overset{y=5^x}{\iff} y^2 - 4y - 5 \geqslant 0, \iff y \leqslant \frac{4 - \sqrt{36}}{2} = -1, \vee\ y \geqslant \frac{4 + \sqrt{36}}{2} = 5,$$

iff $5^x \leqslant -1$ or $5^x \geqslant 5$. The first has not solutions. The second, being $5 > 1$, produce $5^x \geqslant 5 = 5^1$ iff $x \geqslant 1$. Therefore: the solutions are $x \geqslant 1$. $\square$

We claim that the exponential $\mathbb{R} \ni \exp_a : x \mapsto a^x \in [0, +\infty[$ with base $a \neq 1$ is invertible. We know that this is equivalent to say that it is bijective. We already know that it is injective, for it is strictly increasing [resp. decreasing] as soon as $a > 1$ [resp. $a < 1$]. Moreover, it is surjective, as stated in the following result (of which we skip the non trivial proof):

**Theorem 2.4.10**

Let $a > 0$, $a \neq 1$. Then the exponential map $\exp_a : \mathbb{R} \to [0, +\infty[$ $\exp_a(x) = a^x$ is surjective, namely,

$$\forall y > 0, \ \exists x \in \mathbb{R} \ : \ a^x = y.$$

Therefore:

**Theorem 2.4.11**

If $a > 0, a \neq 1$, the function $\exp_a$ is bijective, namely ,

$$\forall y \in ]0, +\infty[, \ \exists! x \in \mathbb{R} \ : \ a^x = y. \quad [3]$$

and such $x$ is called *the logarithm in base $a$ of $y$* and is denoted by $\log_a y := x$.
Namely, the **logarithm function** $\log_a : ]0, +\infty[ \longrightarrow \mathbb{R}$ **is the inverse of** $\exp_a$, that is

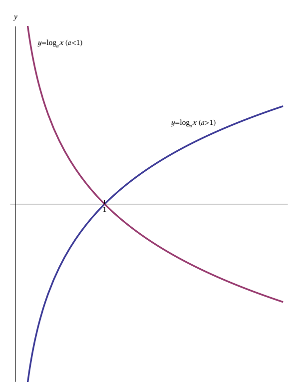
$$\log_a(a^y) = y, \ \forall y \in \mathbb{R}, , \quad a^{\log_a x} = x, \ \forall x \in ]0, +\infty[.$$

i.e.

$$\log_a \circ \exp_a = id_{\mathbb{R}} \qquad \exp_a \circ \log_a = id_{]0,+\infty[}.$$

In particular, the function $\log_a$ it fulfills the following properties:

i) $\log_a \nearrow$ strictly if $a > 1$, $\log_a \searrow$ strictly if $a < 1$.
ii) $\log_a(xy) = \log_a x + \log_a y$, for any $x, y \in ]0, +\infty[$.
iii) $\log_a(x^y) = y \log_a x$, for any $y \in \mathbb{R}$, $x > 0$.
iv) $\log_a x = (\log_a b)(\log_b x)$, for any $x > 0$, $a, b \neq 1$.
iv) $\log_a 1 = 0$, for any $a > 0$, $a \neq 1$.
v) $\log_a a = 1$, for any $a > 0$, $a \neq 1$.




[3]The notation $\exists!$ means "the exists unique". Clearly, here existence comes from surjectivity and uniqueness comes from injectivity.

In solving inequalities it is useful to notice that from the monotonicity of $exp_a$ and $\log_a$ one gets:

$$a^x \geqslant y, \iff \begin{cases} \forall x, & \text{if } y \leqslant 0, \\ \forall x \geqslant \log_a y, & \text{if } y > 0,\ (a > 1), \\ \forall x \leqslant \log_a y, & \text{if } y > 0,\ (a < 1). \end{cases}$$

EXAMPLE 2.4.12. *Solve*

$$\log_2 \sqrt{x-1} + 2 \geqslant \log_4(x+4).\ (\star)$$

SOL. — Let us first discuss about the domain of existence of the inequality. We need

$$\begin{cases} x - 1 \geqslant 0, \\ \sqrt{x-1} > 0, \\ x + 4 > 0, \end{cases} \iff \begin{cases} x \geqslant 1, \\ x > 1, \\ x > -4, \end{cases} \iff x > 1,$$

Therefore the domain is $D = ]1, +\infty[$. Now, by properties of logarithms,

$$(\star) \iff \log_2 \sqrt{x-1} + 2 \geqslant (\log_4 2)(\log_2(x+4)).$$

Clearly $\log_4 2 = \log_4 4^{1/2} = \frac{1}{2}\log_4 4 = \frac{1}{2}$, so

$$(\star) \iff \log_2 \sqrt{x-1} + 2 \leqslant \log_2 \sqrt{x+4}, \iff \log_2 \frac{\sqrt{x-1}}{\sqrt{x+4}} \leqslant -2.\ (\star^2).$$

Being the base $2 > 1$, by monotonicity,

$$(\star^2) \iff \frac{\sqrt{x-1}}{\sqrt{x+4}} \leqslant 2^{-2} = \frac{1}{4}. \iff 4\sqrt{x-1} \leqslant \sqrt{x+4}, \iff 16(x-1) \leqslant x+4, \iff x \leqslant \frac{4}{3}.$$

Hence, the solutions (in $D$) are the interval $]1, \frac{4}{3}]$. □

**2.4.3. Periodicity and simmetry of functions.** Let us introduce the definition of periodic function:

**Definition 2.4.13**

Let $A$ be a (unbounded) subset of $\mathbb{R}$. A function $f : A \longrightarrow \mathbb{R}$ is called $T$**−periodic** if, for every $x \in A$, $x + T \in A$, and

$$f(x+T) = f(x),\ \forall x \in A.$$

Trigonometric functions, which are introduced below, are classical examples of periodic functions. Simmetric and antisimmetric functions (with respect to the origin) are also called *even* and *odd* functions, respectively. More precisely,

**Definition 2.4.14: Even and odd functions**

Let $A \subset \mathbb{R}$ be a subset such that $x \in A \iff -x \in A$.
A function $f : A \longrightarrow \mathbb{R}$ is said to be **even** [resp. **odd** ] if, for every $x \in A$,

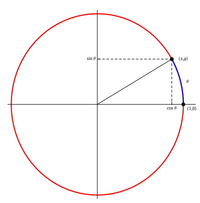
$$f(x) = f(-x) \qquad [\text{resp.} f(x) = -f(-x)]$$

For instance, for every $n \in \mathbb{N}$, the functions $x^{2n}$ and $x^{2n+1}$ (are defined on $\mathbb{R}$ and) are even and odd, respectively.

**2.4.4. Trigonometric functions.** Trigonometric functions are called also *circular functions* because of their geometric meaning. Consider a point $(x, y)$ on the *unitary circumference* of equation

$$x^2 + y^2 = 1.$$

Instead of cartesian coordinates $(x, y)$ we could characterize a point on the circumference by the length $\theta$ of the arc of circumference joining $(1, 0)$ to $(x, y)$ counterclockwise.The total length of the circumference is $2\pi$, where $\pi$ is an irrational number (i.e.belonging to $\mathbb{R}\backslash\mathbb{Q}$), so that it can be approximated as much as we wish with rational numbers. For instance, it is well known that $\frac{314}{100} < \pi < \frac{315}{100}$. Hence, we have $\theta \in [0, 2\pi]$ with $\theta = 0$ corresponding to $(1, 0)$, $\theta = \pi$ (half-circumference) corresponding to $(-1, 0)$, $\theta = \frac{\pi}{2}$ corresponding to $(0, 1)$ and so on. In general, to any point $(x, y)$ on the circumference it corresponds a $\theta \in [0, 2\pi[$. With this convention, given $\theta$ we call the cartesian coordinates $x$ and $y$ as the *sine of* $\theta$ and the *cosine of* $\theta$, respectively, and we write $x = \cos\theta$ and $y = \sin\theta$).




In this way, we have defined two functions $\sin, \cos : [0, 2\pi[ \longrightarrow \mathbb{R}$. For several practical reasons, it is convenient to define sin and cos for any $\theta \in \mathbb{R}$. There is a natural way to proceed. Imagine a string fixed in $(1, 0)$ winding on the circumference. If the length $2\pi$ represent the entire circumference, $2 \cdot 2\pi = 4\pi$ could be like two complete counterclockwise windings on the circumference. In this way, $2\pi + \theta_0$ with $\theta_0 \in [0, 2\pi[$ represent the same point of $\theta_0$. At the same time, a length $-2\pi$ could represent a complete clockwise winding, so $-2\pi + \theta_0$ (again $\theta_0 \in [0, 2\pi[$) represents always the same point as $2\pi + \theta_0$ and $\theta_0$. More in general

$$\theta + k2\pi, \quad k \in \mathbb{Z},$$

is always the same point on the cartesian plane. Therefore, it is reasonable to set

$$\sin(\theta + k2\pi) := \sin\theta, \quad \cos(\theta + k2\pi) := \cos\theta, \quad [0, 2\pi[\forall\theta \in \forall k \in \mathbb{Z}.$$

This is resumed that the functions sin and cos are defined on $\mathbb{R}$ and are $2\pi$-periodic. Some properties of sin and cos are summarized in the following theorem:

**Theorem 2.4.15**

The functions $\sin, \cos : \mathbb{R} \longrightarrow \mathbb{R}$ verify the following properties:

i) $(\cos 0, \sin 0) = (1, 0)$, $\left(\cos \frac{\pi}{2}, \sin \frac{\pi}{2}\right) = (0, 1)$.

ii) (**fundamental identity**): it holds

$$(\sin\theta)^2 + (\cos\theta)^2 = 1, \ \forall \theta \in \mathbb{R}.$$

In particular $-1 \leqslant \sin\theta \leqslant 1$, $-1 \leqslant \cos\theta \leqslant 1$, $\forall\theta \in \mathbb{R}$.

iii) ($2\pi$**-periodicity**): $\sin(\theta + 2\pi) = \sin\theta$, $\cos(\theta + 2\pi) = \cos\theta$, for any $\theta \in \mathbb{R}$.

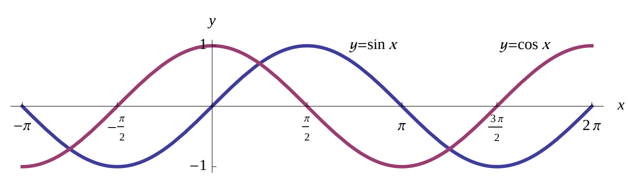
iv) (**addition formulas**)

$$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2, \quad \cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2.$$

In particular: $\sin(2\theta) = 2\sin\theta\cos\theta$, $\cos(2\theta) = (\cos\theta)^2 - (\sin\theta)^2 = 2(\cos\theta)^2 - 1$

v) (**simmetries**): sin and cos are even and odd functions, respectively, that is, $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$ for any $\theta \in \mathbb{R}$.

The addition formula and other similar formulas are usually known from high schools. Their proofs are based on the geometric meaning of $\sin, \cos$ and on triangles similarities.



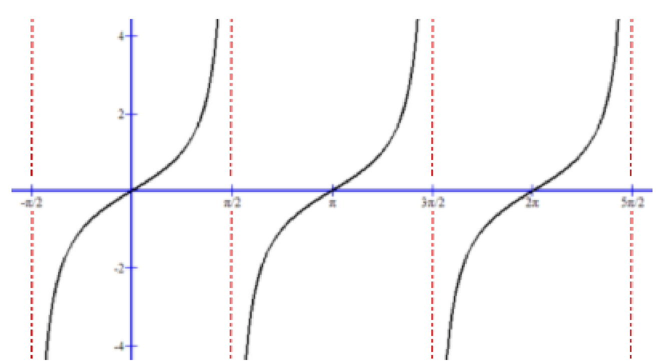

Starting from sin and cos other important functions are defined. For example, the **tangent** tan and the **cotangent** cot are defined as

$$\tan : \mathbb{R} \backslash \left\{ \frac{\pi}{2} + k\pi \ : \ k \in \mathbb{Z} \right\} \longrightarrow \mathbb{R}, \quad \tan\theta := \frac{\sin\theta}{\cos\theta},$$

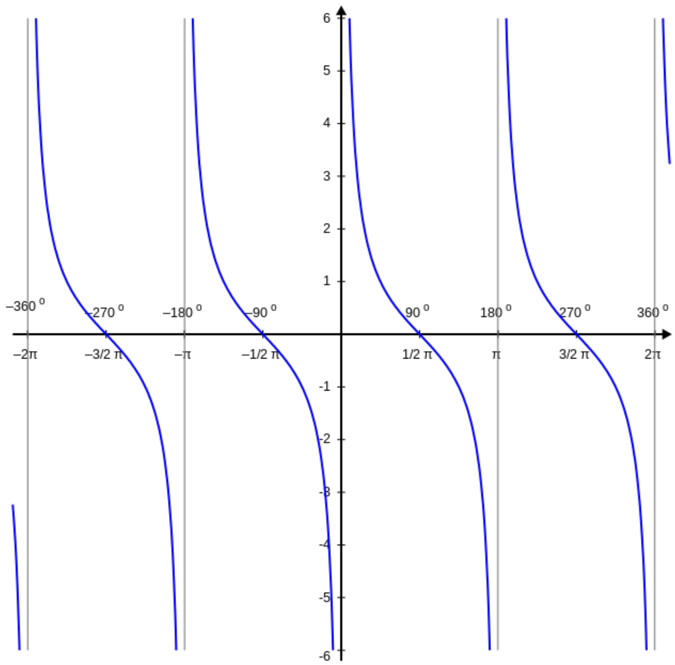
and

$$\cot : \mathbb{R} \backslash \left\{ k\pi \ : \ k \in \mathbb{Z} \right\} \longrightarrow \mathbb{R}, \quad \cot\theta := \frac{\cos\theta}{\sin\theta} \ \left( = \frac{1}{\tan\theta} \right),$$

respectively. It is trivial to check that both tan and cot are $\pi-$periodic.



tangent's graph




cotangent's graph

**2.4.5. Modulus.** The set of real numbers $\mathbb{R}$ is usually represented by a straight line $\ell$ (called "real line") as follows:i) first one chooses a point $O \in \ell$, the *origin*, representing the number 0; ii) secondly one fixes a unit of measure; iii) one choose a positive direction on the line $\ell$; iv) finally, for every point $P \in \ell$, one associates the real number $x$, which will be positive [resp. negative ] provided $P$ is after [resp. before ] the origin $O$ at a distance $|x|$ from the origin $O$, where

$$|x| := \begin{cases} x, & \text{if } x \geqslant 0, \\ \\ -x, & \text{if } x < 0. \end{cases}$$

This gives a geometric representation of numbers. [4] The function $|\cdot| : \mathbb{R} \longrightarrow [0, +\infty[$ defined above is called **modulus**. Clearly $|x| \geqslant 0$ for any $x \in \mathbb{R}$. Here are the fundamental properties of the modulus:

**Proposition 2.4.16**

The following properties hold true

- **vanishing**: $|x| = 0$ iff $x = 0$.
- **homogeneity**: $|xy| = |x||y|$, for any $x, y \in \mathbb{R}$.
- **triangular inequality**: $|x + y| \leqslant |x| + |y|$, for any $x, y \in \mathbb{R}$.

---

[4] Geometry means *measure science on the Earth*,

Proof. The first two are easy exercises. About the third,

$$\begin{array}{l} -|x| \leqslant x \leqslant |x|, \\ -|y| \leqslant y \leqslant |y|, \end{array} \quad \overset{summing}{\Longrightarrow} \quad -(|x|+|y|) \leqslant x+y \leqslant |x|+|y|.$$

From this, the conclusion is evident. □

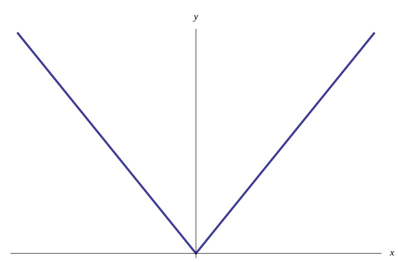
Notice again that

$$|x| = a, \iff \begin{cases} \text{never,} & \text{if } a < 0, \\ x = 0, & \text{if } a = 0, \\ x = \pm a, & \text{if } a > 0. \end{cases}$$

Similarly, when $a > 0$,

$$|x| \leqslant a, \iff -a \leqslant x \leqslant a, \qquad |x| \geqslant a, \iff x \leqslant -a, \vee x \geqslant a.$$

Here's the graph of modulus




Example 2.4.17. *Solve*

$$||x|+1| \leqslant \sqrt{x+2}.$$

Sol. — The domain of existence of the inequality is $\{x \in \mathbb{R} : x+2 \geqslant 0\} = [-2, +\infty[$. Being both members of the inequality positive, we can square,

$$\begin{aligned} ||x|+1| \leqslant \sqrt{x+2}, \quad &\iff ||x|+1|^2 \leqslant x+2, \iff (|x|+1)^2 \leqslant x+2, \\ &\iff |x|^2 + 2|x| + 1 \leqslant x+2, \\ &\iff \begin{cases} x^2+x-1 \leqslant 0, & x \geqslant 0, \\ x^2-3x-1 \leqslant 0, & x < 0. \end{cases} \end{aligned}$$

Now

$$\text{as } x \geqslant 0,\ x^2+x-1 \leqslant 0, \iff \frac{-1-\sqrt{5}}{2} \leqslant x \leqslant \frac{-1+\sqrt{5}}{2},$$

and because $x \geqslant 0$, it follows $x \in \left[0, \frac{-1+\sqrt{5}}{2}\right]$. Moreover,

$$\text{as } -2 \leqslant x < 0,\ x^2-3x-1 \leqslant 0, \iff \frac{3-\sqrt{13}}{2} \leqslant x \leqslant \frac{3+\sqrt{13}}{2},$$

that is, being $x < 0$, $-2 < \frac{3-\sqrt{13}}{2} \leqslant x < 0$. So, solutions are $\left[\frac{3-\sqrt{13}}{2}, \frac{-1+\sqrt{5}}{2}\right]$. □

## 2.5. Density of $\mathbb{Q}$ in $\mathbb{R}$

As we said, $\mathbb{R}$ arises as "extension" of rationals, thus $\mathbb{R}$ is "bigger" than $\mathbb{Q}$. However, no matter what real number we consider, arbitrarily close to it, we may find rational numbers. In some sense, rationals are everywhere sparse on $\mathbb{R}$:

**Theorem 2.5.1**

$$\forall [a,b] \subset \mathbb{R},\ \exists q \in \mathbb{Q} \ :\ q \in [a,b].$$

Proof. By Archimedean property, there exists an integer such that $n \geqslant \frac{1}{b-a}$, so

$$\exists n \in \mathbb{N} \ :\ \frac{1}{n} \leqslant b-a.$$

To proceed with the argument, we assume $a \geqslant 0$ (slight adjustment is needed for $a < 0$). We will show that

$$\exists m \in \mathbb{N},\ :\ a \leqslant m\frac{1}{n} \leqslant b.$$

Indeed: consider the numbers

$$0,\ \frac{1}{n},\ \frac{2}{n},\ \frac{3}{n},\ \ldots,\ \frac{m}{n},\ \ldots$$

Sooner or later $\frac{m}{n} > a$. Indeed, if this is false, that is, if $\frac{m}{n} \leqslant a$ then $m \leqslant na$ for any $m \in \mathbb{N}$ and this would be contrary to the Archimedean property again. Let us take the smallest $m$ such that $\frac{m}{n} > a$. Then

$$\frac{m-1}{n} < a \leqslant \frac{m}{n}.$$

We say that our $m$ is such that $a \leqslant \frac{m}{n} \leqslant b$. Indeed

$$\frac{m}{n} = \frac{m-1+1}{n} = \frac{m-1}{n} + \frac{1}{n} < a + \frac{1}{n} \leqslant a + (b-a) = b. \qquad \square$$

**Definition 2.5.2**

We say that $S \subset \mathbb{R}$ is **dense** in $\mathbb{R}$ if

$$\forall [a,b] \subset \mathbb{R},\ [a,b] \cap S \neq \emptyset.$$

Thus, in particular, $\mathbb{Q}$ is *dense* in $\mathbb{R}$. Similarly, we have

**Theorem 2.5.3**

$\mathbb{R} \backslash \mathbb{Q}$ is dense in $\mathbb{R}$.

## 2.6. Exercise

Exercise 2.6.1. *Show that* $\sqrt{3} \notin \mathbb{Q}$*. In general, can you prove that, for* $n \in \mathbb{N}$*,* $\sqrt{n} \notin \mathbb{Q}$ *unless* $n = m^2$ *for some* $m \in \mathbb{N}$*?*

Exercise 2.6.2. *Solve*

1. $\sqrt{x^2 + x - 1} \geqslant 1$.
2. $\sqrt{|x| + 1} > x - 1$.
3. $|x(x-1)| \leqslant x - 1$.
4. $2^{2x} + 2^{x-1} \geqslant 3$.
5. $\dfrac{1}{2^x} - 2^x \geqslant 1$.
6. $2^x + 2^{-x} \leqslant 2$.
7. $\log_2(x^2) + \log_2(2x) \geqslant 0$.
8. $\log_2 |x+1| + \log_2 |x-3| < 1$.
9. $2^{|x-1|} < 4^{-x}$.

Exercise 2.6.3. *Find the domain of the following functions.*

1. $\sqrt{\log_2 x + \dfrac{2}{3}}$.
2. $\sqrt{2^{|x-3|} - 8} + \sqrt{3^{x^2+x+2} - 9}$.
3. $\log_{10}\left(2^x - \dfrac{1}{8}\right)$.
4. $\log_2\left(2 - \dfrac{\log_2(x-2)}{\sqrt{1 + \log_2(x-2)}}\right)$.
5. $2^{\sqrt{\frac{x^2-4}{2x^2-5x+3}}}$.
6. $\log_4 \sqrt{\dfrac{x+1}{x-1}}$.

Exercise 2.6.4. *(1) Let* $S := \left\{\frac{n}{\sqrt{n^2+1}} \ : \ n \in \mathbb{N},\ n \geqslant 1\right\}$*, show that* $\min S = \frac{1}{\sqrt{2}}$ *and* $\sup S = 1$*. What about* max*?*

(2) *Let* $S := \left\{\frac{n}{\sqrt{n^2-1}} \ : \ n \in \mathbb{Z},\ n \leqslant -2\right\}$*, show that* $\min S = -\frac{2}{\sqrt{3}}$ *and* $\sup S = -1$*. What about* max*?*
(3) *Let* $S := \{n - \sqrt{n-3} \ : \ n \in \mathbb{N},\ n \geqslant 3\}$*, show that* $\min S = 3$ *and* $\sup S = +\infty$*.*
(4) *Let* $S := \left\{3 + \frac{\sqrt{2}}{\sqrt{n^2+7n} - \sqrt{2}} \ : \ n \in \mathbb{N},\ n > 0\right\}$*, show that* $\inf S = 3$ *and* $\max S = 4$*. What about* min*?*
(5) *Let* $S := \{-n + \sqrt{n-2} \ : \ n \in \mathbb{N},\ n \geqslant 2\}$*, show that* $\max S = -2$ *and* $\inf S = -\infty$*.*
(6) *Let* $S := \left\{2 - \frac{\sqrt{2}}{\sqrt{n^2+2n} - \sqrt{2}} \ : \ n \in \mathbb{N},\ n > 0\right\}$*, show that* $\min S = 1$ *and* $\sup S = 2$*. What about* max*?*
(7) *Let* $S := \left\{\frac{\sqrt{n+1}}{\sqrt{n}+1} \ : \ n \in \mathbb{N},\ n \geqslant 1\right\}$*, show that* $\min S = \frac{\sqrt{2}}{2}$ *and* $\sup S = 1$*. What about* max*?*

Exercise 2.6.5. *For any of the following sets, discuss* inf, min, sup, max.

1. $\left\{\dfrac{n-1}{n}\ :\ n \in \mathbb{N}\setminus\{0\}\right\}$ $[0, 0, 1, \nexists]$

2. $\left\{\dfrac{n+\sqrt{n}}{n-\sqrt{n}}\ :\ n \in \mathbb{N},\ n > 1\right\}$ $[1, \nexists, 1, \frac{2+\sqrt{2}}{2-\sqrt{2}}]$

3. $\left\{1+\sqrt{\dfrac{n}{n+1}}\ :\ n \in \mathbb{N}\right\}$ $[1, 1, 2, \nexists]$

4. $\left\{\dfrac{2n^2+1}{2n^2+2n+1}\ :\ n \in \mathbb{N}\setminus\{0\}\right\}$ $[\frac{3}{5}, \frac{3}{5}, 1, \nexists]$

5. $\left\{\dfrac{n^2+3n+4}{n^2+3n+3}\ :\ n \in \mathbb{N}\right\}$ $[1, \nexists, \frac{4}{3}, \frac{4}{3}]$

6. $\left\{\dfrac{\sqrt{n}-1}{\sqrt{n}+1}\ :\ n \in \mathbb{N},\ n \geqslant 1\right\}$ $[0, 0, 1, \nexists]$

7. $\left\{\log_2\left(1+\sqrt{\dfrac{n}{n+1}}\right)\ :\ n \in \mathbb{N}\right\}$ $[0, 0, 1, \nexists]$

8. $\left\{\dfrac{\sqrt{1+3^{2n}}}{3^n}\ :\ n \in \mathbb{N}\right\}$ $[1, \nexists, \sqrt{2}, \sqrt{2}]$

9. $\left\{\dfrac{2^n}{\sqrt{4^n-1}}\ :\ n \in \mathbb{N}\setminus\{0\}\right\}$ $[1, \nexists, \frac{2}{\sqrt{3}}\ \frac{2}{\sqrt{3}}]$

10. $\left\{\log_{1/2}\sqrt{\dfrac{n+1}{n}}\ :\ n \in \mathbb{N}\setminus\{0\}\right\}$ $[-\frac{1}{2}, -\frac{1}{2}, 0, \nexists]$

11. $\left\{\dfrac{\log_2 n}{\sqrt{1+(\log_2 n)^2}}\ :\ n \in \mathbb{N}\setminus\{0\}\right\}$ $[0, 0, 1, \nexists]$

12. $\left\{\dfrac{4^n-1}{4^n+2^n+1}\ :\ n \in \mathbb{Z}\right\}$ $[-1, \nexists, 1, \nexists]$

13. $\left\{(-1)^n\dfrac{n}{n+1}\ :\ n \in \mathbb{N}\right\}$ $[-1, \nexists, 1, \nexists]$

14. $\left\{2^{\frac{(-1)^n n^2}{n+1}}\ :\ n \in \mathbb{N}\right\}$ $[0, \nexists, +\infty, \nexists]$

15. $\left\{2^{(-1)^n n}\sin\left(\dfrac{n\pi}{3}\right)\ :\ n \in \mathbb{N}\right\}$ $[-\infty, \nexists, +\infty, \nexists]$

16. $\left\{\dfrac{n-1}{n+1}\cos\dfrac{n\pi}{2}\ :\ n \in \mathbb{N}\right\}$ $[-1, \nexists, 1, \nexists]$

CHAPTER 3

# Complex Numbers

## 3.1. Why do we need Complex Numbers?

Most of the topics of this course are based on real numbers. There is, however, a certain number of questions that demand for a further extension of the concept of number. Since this extension is purely algebraic and of great relevance in applications, we will spend this chapter on it.

To motivate *complex numbers*, we consider an ancient algebraic problem: *to solve any polynomial equation* like

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0.$$

A *solution* of this equation is a value of $x$ which makes the above equality true. For every $n \geq 1$, we say that this equation *has* $n - th$ *degree if* $c_n \neq 0$. The case of first degree, $c_1 x + c_0 = 0$, is straightforward:

$$c_1 x + c_0 = 0 \Longleftrightarrow x = -\frac{c_0}{c_n},$$

i.e., the equation has one and only one solution, namely $-\frac{c_0}{c_n}$. The case $n = 2$, namely, the second degree equation,

$$\alpha x^2 + \beta x + \gamma = 0 \qquad (\alpha \neq 0)$$

is less trivial (for notational simplicity we write $\alpha, \beta, \gamma$ instead of $c_1, c_2, c_3$, rispectively). A special case of this type of equations is, for example, $x^2 = k$ whose solutions are, if any, the square roots of $k$. Since we know that only positive (or zero) numbers have square roots in $\mathbb{R}$, we may conclude that *in general, a second degree equation is not solvable in* $\mathbb{R}$ [1].

Everybody has seen, during high school, that real solutions exist if and only if the *discriminant* $\Delta := \beta^2 - 4\alpha\gamma$ is $\geq 0$. In that case there are 2 solutions $x_1, x_2$ given by the formula

$$x_1 = \frac{-\beta - \sqrt{\Delta}}{2\alpha} \quad x_2 = \frac{-\beta + \sqrt{\Delta}}{2\alpha}$$

Though elementary, the method to get this formula is often not known by students. Here it is: we first rewrite the equation as $x^2 + \frac{\beta}{\alpha}x + \frac{\gamma}{\alpha} = 0$; then, we reconstruct the structure of a square by writing the equation as $x^2 + 2\frac{\beta}{2\alpha}x + \left(\frac{\beta}{2\alpha}\right)^2 = \left(\frac{\beta}{2\alpha}\right)^2 - \frac{\gamma}{\alpha}$, that is, $\left(x + \frac{\beta}{2\alpha}\right)^2 = \frac{\beta^2 - 4\alpha\gamma}{4\alpha^2} = \frac{\Delta}{4\alpha^2}$. Thus, provided $\Delta \geq 0$, one obtains $x + \frac{\beta}{2\alpha} = \frac{\sqrt{\Delta}}{2\alpha}$ or $x + \frac{\beta}{2\alpha} = -\frac{\sqrt{\Delta}}{2\alpha}$, which in turn give the solutions $x_1$ and $x_2$ above.

[1] Let us recall that ”in general” means ”sometimes yes, but sometimes no”: in this case, for instance, $x^2 + 1$ is not solvable.

Now we want to see what can be done if $\Delta < 0$. No solution exists in $\mathbb{R}$, and the previous formulas loose any meaning. However, imagine for a moment $\Delta < 0$ and, to emphasize this, we write $\Delta = -1(-\Delta)$. Then, *with an abuse of notation and acting formally*, we may write

$$x = \frac{-\beta \pm \sqrt{-1(-\Delta)}}{2\alpha} = \frac{-\beta \pm \sqrt{-1}\sqrt{-\Delta}}{2\alpha}.$$

Of course $\sqrt{-1}$ is out of meaning in $\mathbb{R}$, but we may look at it as a "new number" in an enlarged set of numbers. [2] This new number is usually named *imaginary unit* and the symbol $i := \sqrt{-1}$ is used. This way, the solutions of the second degree equation would have the form

$$x = -\frac{\beta}{2\alpha} \pm i\frac{\sqrt{-\Delta}}{2\alpha} = a + ib, \qquad a := -\frac{\beta}{2\alpha}, \quad : b = \frac{\sqrt{-\Delta}}{2\alpha}$$

The "numbers" of the form $a + ib$, with $z := a, b \in \mathbb{R}$, are called *complex numbers*. Of course, to be true numbers and not just fancy notations, we need to structure an algebra on them. This is not only possible, but simple and natural.

Complex numbers are not only true numbers and interesting to give meaning to solutions of a second degree equation. We will see here that roots always exist (De Moivre theorem). It is possible to show that any algebraic (polynomial) equation has always solutions in complex numbers (fundamental theorem of Algebra). Complex numbers have not just a mathematical interest: they are widely used in many contexts, from physics to engineering.

### 3.2. Definition of $\mathbb{C}$

We introduced new numbers of the form

$$a + ib,$$

where $a, b \in \mathbb{R}$ and $i$ having the meaning of *a square of* $-1$, that is $i^2 = -1$. To consider them as *numbers*, we need to define operations. These are quite natural because are based on the formal algebraic calculus:

$$(a + ib) + (c + id) = (a + c) + i(b + d),$$

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc).$$

Now, the program is to take the previous relations as the definition of the sum and product of complex numbers. We will show that these operations fulfil algebraic properties at all similar to those verified by the sum and product of real numbers. The main difference with $\mathbb{R}$ is that, as we will see, on $\mathbb{C}$ it is not possible to have an ordering of numbers.

---

[2] In some sense, it is like $\sqrt{2}$ for $\mathbb{Q}$: $\sqrt{2}$ does not have any meaning in $\mathbb{Q}$ but it makes sense in a larger set, namely in $\mathbb{R}$.

**Definition 3.2.1**

We call **set of complex numbers**

$$\mathbb{C} := \{a + ib \ : \ a, b \in \mathbb{R}\}.$$

Given $z = a + ib$, $a, b \in \mathbb{R}$,

- $a$ is called **real part of** $z$ (notation $\operatorname{Re} z$),
- $b$ is called **imaginary part of** $z$ (notation $\operatorname{Im} z$).

On $\mathbb{C}$ the following operations are defined:

- **sum**
$$(a + ib) + (c + id) := (a + c) + i(b + d), \quad \forall a, b, c, d \in \mathbb{R}.$$
- **product**
$$(a + ib)(c + id) := (ac - bd) + i(ad + bc), \quad \forall a, b, c, d \in \mathbb{R}$$

Observe that one needs not remember this formulas by heart, since they can be recovered by just proceeding formally and regarding $i$ as a new number such that $i^2 = -1$.

Example 3.2.2. We actually highly recommend to proceed formally even in simple expressions. For instance,

- $(1 + i2) + (3 + i4) = (1 + 3) + i(2 + 4) = 4 + i6$.
- $(1 + i2)(3 + i4) = 3 + i6 + i8 + i^2 8 = 3 - 8 + i14 = -5 + i14$.
- $(1 + i)^2 = 1 + i^2 + 2i = 1 - 1 + i2 = 0 + i2$. □

Sums and products of complex numbers fulfil similar properties to their homologous on $\mathbb{R}$:

**Theorem 3.2.3**

On $\mathbb{C}$ the following properties of sum and product hold true:

- **sum**:
  i) (**associativity**): $z + (w + \zeta) = (z + w) + \zeta$, $\forall z, w, \zeta \in \mathbb{C}$.
  ii) (**commutativity**): $z + w = w + z$, $\forall z, w \in \mathbb{C}$.
  iii) (**zero**): $z + 0_{\mathbb{C}} = z$, $\forall z \in \mathbb{C}$, where $0_{\mathbb{C}} = 0 + i0$.
  iv) (**opposite**): $\forall z \in \mathbb{C}$, the opposite of $z = a + ib$ is $-z = (-a) + i(-b)$. We write $-z = -a - ib$.
- **product**:
  i) (**associativity**): $z \cdot (w \cdot \zeta) = (z \cdot w) \cdot \zeta$, $\forall z, w, \zeta \in \mathbb{C}$.
  ii) (**commutativity**): $z \cdot w = w \cdot z$, $\forall z, w \in \mathbb{C}$.
  iii) (**unit**): $z \cdot 1_{\mathbb{C}} = z$, $\forall z \in \mathbb{C}$ where $1_{\mathbb{C}} = 1 + i0$.
  iv) (**reciprocal**): $\forall z \in \mathbb{C} \backslash \{0_{\mathbb{C}}\}$, $\exists! w \in \mathbb{C} : z \cdot w = 1_{\mathbb{C}}$. We write $\frac{1}{z} := w$.
- **distributivity**: $w \cdot (z + \zeta) = w \cdot z + w \cdot \zeta$, $\forall w, z, \zeta \in \mathbb{R}$.

Proof. It is easy, we will limit to a few properties as examples.

**Sum —** i) Exercise.
ii) Let

$$z = a + ib, \; w = c + id.$$

Then

$$z + w = (a + c) + i(b + d) = (c + a) + i(d + b) = w + z.$$

iii) If $z = a + ib$ then

$$z + 0_{\mathbb{C}} = (a + ib) + (0 + i0) = (a + 0) + i(b + 0) = a + ib = z.$$

iv) Exercise.

**Product —** i), ii) Exercise
iii) If $z = a + ib$ then

$$z1_{\mathbb{C}} = (a + ib)(1 + i0) = a + ib + ia \cdot 0 + i^2 b \cdot 0 = a + ib = z.$$

iv) Let $z = a + ib \neq 0_{\mathbb{C}}$. Then

$$\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{a-ib}{(a+ib)(a-ib)}.$$

Now,

$$(a + ib)(a - ib) = a^2 - i^2 b^2 + iba - iab = a^2 + b^2.$$

Since $z = a + ib \neq 0_{\mathbb{C}} = 0 + i0$, $a, b$ cannot be both zero, thus $a^2 + b^2 > 0$. Therefore,

$$\frac{1}{z} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} + i\left(-\frac{b}{a^2+b^2}\right).$$

**Distributivity:** exercise. □

**Definition 3.2.4**

Given $z = a + ib$ the number $\bar{z} = a - ib$ is called **complex conjugate** of $z$.

Immediate simple properties you may easily verify as an exercise are

i) $\overline{z + w} = \bar{z} + \bar{w}$, $\forall z, w \in \mathbb{C}$.
ii) $\overline{zw} = \bar{z}\bar{w}$, $\forall z, w \in \mathbb{C}$.
iii) $z = \bar{z}$ iff $\operatorname{Im} z = 0$.
iv) $z + \bar{z} = 2\operatorname{Re} z$, $z - \bar{z} = i2\operatorname{Im} z$, $\forall z \in \mathbb{C}$
v) $z \cdot \bar{z} = a^2 + b^2$ (in particular, $z \cdot \bar{z}$ is a non-negative real number).

Observe that in order to calculate the inverse $\frac{1}{z}$ we have used the trick of multiplying both numerator and denominator of $\frac{1}{z}$ by the conjugate $\bar{z}$. Actually, the trick of multiplying numerator and denominator by the conjugate of the denominator makes the division quite simple. Indeed, if $c + id \neq 0$ one has

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd) + i(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

Example 3.2.5. *Compute* $\dfrac{1+i2}{3+i4}$.

Sol. —
$$\frac{1+i2}{3+i4} = \frac{(1+i2)(3-i4)}{(3-i4)(3+i4)} = \frac{3-8+i(6+8)}{25} = \frac{-5+i14}{25} = -\frac{1}{5}+i\frac{4}{25}. \qquad \square$$

We introduce also some useful notations:

- Numbers of the form $a+i0$ are called just **real numbers** and we write simply $a+i0 \equiv a$;
- numbers of the form $0+ib$ are called **imaginary** and we write $0+ib \equiv ib$.
- zero and unit of $\mathbb{C}$ are simply denoted by 0 and 1.

We show that the algebraic structure of $\mathbb{C}$ fulfils property at all similar to those of $\mathbb{R}$. On $\mathbb{R}$ there is also a total ordering. We finish this Section proving that in $\mathbb{C}$ this is impossible:

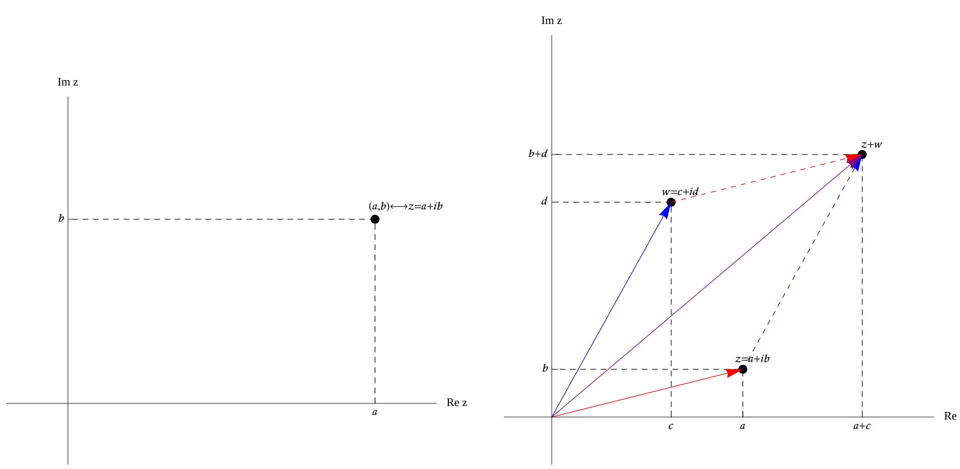
**Proposition 3.2.6**

A total ordering fulfilling invariance with respect to sum and product is not possible on $\mathbb{C}$.

Proof. Take $i$. Since $i \neq 0$, if there would be a total ordering on $\mathbb{C}$, then we should have $i > 0$ or $i < 0$. None of these can be true. In the first case, multiplying by $i > 0$ we would have $i \cdot i > 0$, that is $-1 > 0$. However, then
$$(-1)\cdot(-1) > 0 \cdot (-1), \iff 1 > 0, \iff -1 < 0,$$
which is manifestly impossible together with $-1 > 0$. The same conclusion follows if we assume $i < 0$. $\square$

## 3.3. Gauss plane

As for real numbers, it is convenient to represent a complex number in a geometrical manner. Given $z = a+ib$, $a, b \in \mathbb{R}$ it is natural to associate to $z$ the point $(a,b)$ in the cartesian plane.




The abscissa axis is called *real axis*, the ordinate axis *imaginary axis*. This representation gives an immediate geometrical interpretation to the sum (parallelogram rule). In particular, $\bar{z}$ has a natural geometric interpretation: it is the reflected of $z$ with respect to the real axis. Furthermore, it provides a natural way to define distances between numbers and, in particular, the modulus of a complex number:

**Definition 3.3.1: Modulus**

Let $z = a + ib$, $a, b \in \mathbb{R}$. We posit

$$|z| := \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + b^2}.$$

Remark 3.3.2 (Warning!). A pretty common **error** : $|z| = \sqrt{z^2} = \sqrt{a^2 + (ib)^2} = \sqrt{a^2 - b^2}$. Indeed, $z^2 \neq z \cdot \bar{z}$ in general, and $z^2 = z \cdot \bar{z}$ only for $z$ real, that is $Im(z) = 0$ (prove it by exercise). □

Modulus of complex numbers fulfils the same properties of the modulus of real numbers. Clearly, $|z| \geqslant 0$ for every $z \in \mathbb{C}$. Moreover,

**Proposition 3.3.3**

The following properties hold:

- **vanishing**: $|z| = 0$ iff $z = 0$.
- **homogeneity**: $|zw| = |z||w|$, $\forall z, w \in \mathbb{C}$.
- **triangular inequality**: $|z + w| \leqslant |z| + |w|$, $\forall z, w \in \mathbb{C}$.

Proof. **Vanishing:** $|z| = 0$ iff $\sqrt{a^2 + b^2} = 0$, that is $a^2 + b^2 = 0$, that is again iff $a = b = 0$.

**Homogeneity**: we prove that $|zw|^2 = |z|^2|w|^2$. Once this is achieved, we draw $|zw| = \pm|z||w|$, and being $|zw| \geqslant 0$, necessarily $|zw| = |z||w|$. Thus, let us prove $|zw|^2 = |z|^2|w|^2$. Let $z = a + ib$ and $w = c + id$. Then

$$zw = (a + ib)(c + id) = (ac - bd) + i(ad + bc),$$

so

$$\begin{aligned} |zw|^2 &= (ac - bd)^2 + (ad + bc)^2 = a^2c^2 + b^2d^2 - 2abcd + a^2d^2 + b^2c^2 + 2abcd \\ &= (a^2 + b^2)(c^2 + b^2) = |z|^2|w|^2. \end{aligned}$$

**Triangular inequality:** let $z = a + ib$ and $w = c + id$. Then

$$|z + w|^2 = (a + c)^2 + (b + d)^2 = a^2 + c^2 + 2ac + b^2 + d^2 + 2bd = |z|^2 + |w|^2 + 2(ac + bd).$$

Claim: $ac + bd \leqslant |z||w|$. If this is true, $|z + w|^2 \leqslant |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2$ that is, $|z + w| \leq |z| + |w|$, which is our thesis. To prove the claim, notice that if $ac + bd \leqslant 0$ there is nothing to prove. Otherwise

$$0 < ac + bd \leqslant |z||w|, \iff (ac + bd)^2 \leqslant |z|^2|w|^2, \iff a^2c^2 + b^2d^2 + 2abcd \leqslant (a^2 + b^2)(c^2 + d^2),$$

that is

$$2abcd \leqslant a^2d^2 + b^2c^2.$$

This follows by the inequality $2\alpha\beta \leqslant \alpha^2 + \beta^2$, which holds true in that $0 \leq (\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$ □

Exercise 3.3.4. Prove the interesting relation (called *Cosine Theorem*):

$$|z+w|^2 = |z|^2 + |w|^2 + 2\mathrm{Re}\, z\bar{w} \qquad \forall z, w \in \mathbb{C}$$

Sol. — We have seen that , setting $z = a + ib$, $w = c + id$, one gets

$$|z+w|^2 = |z|^2 + |w|^2 + 2(ac + bd).$$

So, to prove the above equality it is sufficient to show that $\mathrm{Re}\, z\bar{w} = (ac + bd)$. Actually, it is

$$\mathrm{Re}\,(z\bar{w}) = \mathrm{Re}\,\Big((a+ib)(c-id)\Big) = \mathrm{Re}\,\Big((ac+bd) + i(bc-ad)\Big) = ac + bd$$

□

Example 3.3.5. *Solve*

$$z^2 + 2\bar{z} = |z|^2.$$

Sol. — Let $z = x + iy$ (here $x, y \in \mathbb{R}$ are unknown). Then

$$z^2 = (x+iy)^2 = x^2 - y^2 + i2xy, \quad \bar{z} = x - iy, \quad |z|^2 = x^2 + y^2.$$

Thus, the equation becomes

$$x^2 - y^2 + i2xy + 2(x - iy) = x^2 + y^2, \quad \Longleftrightarrow \quad (-2y^2 + 2x) + i(2xy - 2y) = 0.$$

We may look at this identity as a certain complex number $= 0$. Equivalently, we get a system

$$\begin{cases} -2y^2 + 2x = 0, \\ \\ 2xy - 2y = 0, \end{cases} \Longleftrightarrow \begin{cases} x = y^2, \\ \\ y(x-1) = 0. \end{cases} \Longleftrightarrow \begin{cases} y = 0, \\ \\ x = 0, \end{cases} \text{or} \begin{cases} x = 1, \\ \\ y^2 = 1 \end{cases}$$

We conclude that solutions of the original equations are: $0 + i0 \equiv 0$, $1 \pm i$. □

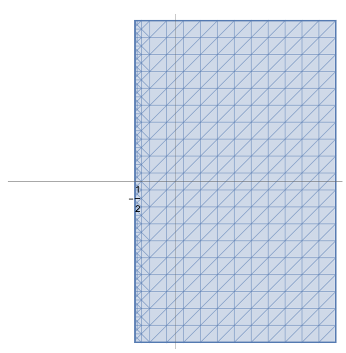
Example 3.3.6. Draw, in the Gauss plane, the set $S := \left\{\left|\frac{z+1}{z}\right| \geqslant 1\right\}$.

Sol. — First $z \neq 0$ when $z \in S$. We may notice that in this case

$$z \in S, \quad \Longleftrightarrow \quad |z+1| \geqslant |z|, \quad \Longleftrightarrow \quad |z+1|^2 \geqslant |z|^2.$$

If $z = x + iy$, we have

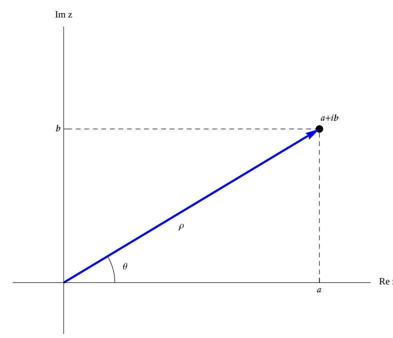
$$(x+1)^2 + y^2 \geqslant x^2 + y^2, \quad \Longleftrightarrow \quad 2x + 1 \geqslant 0, \quad \Longleftrightarrow \quad x \geqslant -\frac{1}{2}. \quad \square$$



### 3.4. Trigonometric and exponential representation of $\mathbb{C}$

There is another natural way to represent a complex number. We may characterize a point $(a, b)$ corresponding to $z = a + ib$ through *polar coordinates* $(\rho, \theta)$ such that

$$\begin{cases} a = \rho \cos\theta, \\ b = \rho \sin\theta. \end{cases}$$




$\rho$ and $theta$ are called the *modulus* and the *argument* of $z$. Clearly $\rho = |z| = \sqrt{a^2 + b^2}$, which justifies the same name of $\rho$ and $z$. As for the argument, $\theta$, one has, for instance:

$$\text{if } a \neq 0 \quad \theta = \arctan\frac{b}{a}, \quad \text{if } a = 0 \text{ and } b < 0, \;\; \theta = \frac{3\pi}{2}, \quad \text{if } a = 0 \text{ and } b > 0, \;\; \theta = \frac{\pi}{2}$$

Thus setting

$$e^{i\theta} := \cos\theta + i\sin\theta, \tag{3.4.1}$$

we get

$$z = a + ib = \rho\,(\cos\theta + i\sin\theta) = \rho e^{i\theta},$$

The expressions $a+ib$, $\rho\,(\cos\theta + i\sin\theta)$, and $\rho e^{i\theta}$ are called the *trigonometric form of* $z$, the *trigonometric form of* $z$, and the *exponential form of* $z$, respectively.

Let us accept (3.4.1) as a mere notation, even though it is actually very well justified by *series expansions*, a subject we will address later in these notes.

Notice that $|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$. Furthermore, since $\sin, \cos$ are periodic functions defined on the entire real line, we may allow $\theta$ to be any real number. This leads to a little ambiguity because

$$\rho e^{i\theta} = \rho e^{i(\theta + k2\pi)}, \;\forall k \in \mathbb{Z}.$$

The ambiguity comes when we have two numbers $z$ and $w$ in trigonometric form. We have

$$\rho e^{i\theta} = r e^{i\phi}, \iff \begin{cases} \rho = r, \\ \theta - \phi = k2\pi, \;\; k \in \mathbb{Z}. \end{cases}$$

Trigonometric and exponential notation is particularly appreciated with products/quotients:

**Proposition 3.4.1**

Let $z = \rho e^{i\theta}$, $w = re^{i\phi}$. Then
i) $zw = \rho r e^{i(\theta+\phi)}$;
ii) if $w \neq 0$ then $\frac{z}{w} = \frac{\rho}{r} e^{i(\theta-\phi)}$;
iii) $z^n = \rho^n e^{in\theta}$, $\forall n \in \mathbb{N}$.

Proof. i) Clearly $zw = r\rho e^{i\theta} e^{i\phi}$. Now, by addition formulas,

$$\begin{aligned} e^{i\theta} e^{i\phi} &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= \cos(\theta+\phi) + i\sin(\theta+\phi) \\ &= e^{i(\theta+\phi)}. \end{aligned}$$

By this the conclusion follows. Other proofs are similar. □

Example 3.4.2. *Compute* $(1+i)^{25}$ *and* $(1+i)^{100}$.

Sol. — We could do the tedious calculations developing the power... or just notice that

$$1 + i = \rho e^{i\theta}, \text{ con } \rho = |1+i| = \sqrt{1^2 + 1^2} = \sqrt{2},\ \theta = \frac{\pi}{4}.$$

Hence

$$(1+i)^{25} = \left(\sqrt{2} e^{\frac{\pi}{4}}\right)^{25} = 2^{25/2} e^{i25\frac{\pi}{4}} = 2^{25/2} e^{i25\frac{\pi}{4}} = 2^{25/2} e^{i\frac{\pi}{4}} == 2^{25/2}(\sqrt{2}/2 + i\sqrt{2}/2) = 2^{13}(1+i).$$

and

$$(1+i)^{100} = \left(\sqrt{2} e^{i\frac{\pi}{4}}\right)^{100} = 2^{100/2} e^{i100\frac{\pi}{4}} = 2^{50} e^{i\pi} = -2^{50}. \quad \square$$

One of the most significant applications of trigonometric notation is to the problem of $n-$th roots of a complex number.

**Theorem 3.4.3: De Moivre**

Every non zero complex number has $n$ different $n-$th roots. Precisely, let $w = \rho e^{i\theta}$ with $\rho > 0$ (that is $w \neq 0$). Then

$$z^n = w, \iff z = \sqrt[n]{\rho}\, e^{i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)},\ k = 0, 1, 2, \dots, n-1.$$

Proof. Assume $z = \varrho e^{i\phi}$ is such that

$$z^n = w, \iff r^n e^{in\phi} = \rho e^{i\theta} \overset{princ.\ id.}{\iff} \begin{cases} r^n = \rho, \\ n\phi = \theta + k2\pi,\ k \in \mathbb{Z}, \end{cases} \iff \begin{cases} r = \rho^{1/n}, \\ \phi = \frac{\theta}{n} + k\frac{2\pi}{n},\ k \in \mathbb{Z}. \end{cases}$$

It is easy to check that for $k = 0, 1, \dots, n-1$ you obtain $n$ different solutions. □

Remark 3.4.4. Of course, the number $w = 0$ ha a unique $n-$th root equal to 0.

Remark 3.4.5. By interpreting geometrically De Moivre theorem we get:

*the $n$ different $n-th$ roots of a complez number $w$ lies on the $n$ vertices of a regular polygon with $n$ angles inscribed in a circle of radius $\sqrt[n]{|w|}$ centered in the origin*

Example 3.4.6. Compute the fourth roots of $i$.

Sol. — First, let's write $i$ in trigonometric notation $\rho e^{i\theta}$. Clearly $\rho = 1$ and $\theta = \frac{\pi}{2}$. Hence

$$z = \varrho e^{i\vartheta},\ z^4 = i, \iff \begin{cases} \varrho = 1^{1/4} = 1, \\ \vartheta = \dfrac{\pi/2}{4} + k\dfrac{2\pi}{4} = \dfrac{\pi}{8} + k\dfrac{\pi}{2}, \end{cases}$$

for $k = 0, 1, 2, 3$. So the four roots are

$$z_1 = cos\left(\frac{\pi}{8}\right)+i\sin\left(\frac{\pi}{8}\right), \quad z_2 = cos\left(\frac{5\pi}{8}\right)+i\sin\left(\frac{5\pi}{8}\right), \quad z_3 = cos\left(\frac{9\pi}{8}\right)+i\sin\left(\frac{9\pi}{8}\right), \quad z_4 = cos\left(\frac{13\pi}{8}\right)+i\sin\left(\frac{13\pi}{8}\right),$$

To write the roots in algebraic form, by means of *bisection formulas*, let us calculate

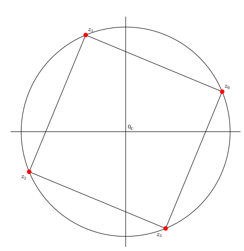
$$\cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{1+\cos\left(\frac{\pi}{4}\right)}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2},$$

$$\sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

By the obvious simmetry we then get

$$z_1 = \frac{\sqrt{2+\sqrt{2}}}{2} + i\frac{\sqrt{2-\sqrt{2}}}{2}, \quad z_2 = -\frac{\sqrt{2-\sqrt{2}}}{2} + i\frac{\sqrt{2+\sqrt{2}}}{2}$$

$$z_3 = -z_1 = -\frac{\sqrt{2+\sqrt{2}}}{2} - i\frac{\sqrt{2-\sqrt{2}}}{2}, \quad z_4 = -z_2 = \frac{\sqrt{2-\sqrt{2}}}{2} - i\frac{\sqrt{2+\sqrt{2}}}{2}$$



Using complex roots we may prove that

**Proposition 3.4.7**

Every second degree equation

$$az^2 + bz + c = 0, \ (a \neq 0, \ a, b, c \in \mathbb{C}),$$

has two complex solutions (coincident if $\Delta = b^2 - 4ac = 0$),

$$z_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a},$$

where $\pm\sqrt{\Delta}$ are the two square roots of complex number $\Delta$. In particular, the following factorization holds:

$$az^2 + bz + c = a(z - z_1)(z - z_2), \ \forall z \in \mathbb{C}.$$

Proof. As in the real case, we may rewrite the equation as

$$\left(z + \frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2}.$$

Thus $z + \frac{b}{2a}$ are the two roots of $\frac{\Delta}{4a^2}$. By De Moivre formula, square roots are always of type $\pm$ one of the roots, and a root of $\frac{\Delta}{4a^2}$ is $\frac{\sqrt{\Delta}}{2a}$. By this conclusion follows. □

Actually a more general result holds, the so-called *Fundamental Theorem of Algebra.* Before stating it, let us recall an elementary result of algebra, sometimes referred as Ruffini's Theorem:

**Theorem 3.4.8**

Let $p(z)$ be a complex polynomial of degree $n$ ($n \geq 1$), and let $w$ be any complex number. Then $w$ is a *root* of $p(z)$, by which one means a solution of the equation $p(z) = 0$

if and only if

$p(z)$ is *divisible by* $(z - w)$, namely there exists a polynomial $q(z)$ (necessarily of degree $n - 1$) such that

$$p(z) = q(z)(z - w) \qquad \forall z \in \mathbb{C}.$$

The proof of the "if" is trivial, while the "only if" part is a mere application of the rule to find the quotient of two polynomials (*Ruffini's rule*).

**Theorem 3.4.9: Fundamental Theorem of Algebra**

Let $n \in \mathbb{N}, \geq 1$. Every $n-$degree polynomial

$$p(z) = \alpha_n z^n + \ldots, +\alpha_1 z + \alpha_0$$

with complex coefficients $\alpha_0, \ldots, \alpha_n$ ($\alpha_n \neq 0$) has $1 \leqslant N \leqslant n$ distinct roots $z_1, \ldots, z_N \in \mathbb{C}$, so that (by Ruffini's Theorem)

$$p(z) = \alpha_n (z - z_1)^{m_1} \cdots (z - z_N)^{m_N}$$

In particular,with $m_1 + \ldots + m_N = n$. for every $k = 1, \ldots, N$, the number $m_k \in \mathbb{N}$ is called **multiplicity of root** $z_k$.

The proof of this theorem relies on continuity arguments which are beyond the level of these notes. The interested reader can find it on most real analysis or complex analysis books. [3]

As a corollary of the Fundamental Theorem of Algebra one can prove the following version of the result in the case of real polynomials (i.e. polynomial with real coefficients). It turns out that real polynomials can be factorized by 1th-degree polynomials and 2nd-degree *real, not factorable*[4] polynomials. **??**

**Theorem 3.4.10: Fundamental Theorem of Algebra f is or real polynomials**

. Let $n \in \mathbb{N}, \geq 1$. Every $n-$degree real polynomial

$$p(x) = a_n x^n + \ldots + a_1 x + a_0,$$

$a_0, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$ can be factorized as follows: there exist

i) two natural numbers $N_1, N_2$, with $0 \leq N_1 + N_2 \leq n$,
ii) positive numbers $\mu_1, \ldots, \mu_{N_1}, \nu_1, \ldots, \nu_{N_2}$
iii) $N_1$ distinct real numbers $x_1, \ldots, x_{N_1}$
iv) $N_2$ distinct real pairs $(b_1, c_1), \ldots, (b_{N_2}, c_{N_2})$ verifying $\Delta_h := b_h^2 - 4c_h < 0$ for all $h = 1, \ldots, N_2$,

such that

$$p(x) = a_n (x - x_1)^{\mu_1} \cdots (x - x_{N_1})^{\mu_{N_1}} (x^2 + b_1 x + c_1)^{\nu_1} \cdots (x^2 + b_{N_2} x + c_{N_2})^{\nu_{N_2}}$$

Remark 3.4.11. In particular, $x_1, \ldots, x_{N_1}$ are the (pairwise different) real roots of $p(x)$, and, for every $k = 1, \ldots, N_1$, $\mu_k$ is the multiplicity of the root $x_k$. Furthermore, for all $h = 1, \ldots, N_2$, $\Delta_h := b_h^2 - 4c_h$ is the discriminant of the 2-degree polynomial $(x^2 + b_h x + c_h)$, and the fact that $\Delta_h$ is negative is equivalent that it cannot be factorized as the product of two *real* 1-degree polynomials. But of course, it can be factorized as the product of two *complex, not real* 1-degree polynomials:

$$(x^2 + b_h x + c_h) = \left(x - \frac{-b + i\sqrt{\Delta_h}}{2}\right)\left(x - \frac{-b - i\sqrt{\Delta_h}}{2}\right),$$

[3]One may also refer to the website https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra.

[4]Let us remind that 2nd-degree not factorable if and only if its discriminant is negative.

and both the roots $\frac{-b+i\sqrt{\Delta_h}}{2}$ and $\frac{-b-i\sqrt{\Delta_h}}{2}$ have multiplicity $\nu_h$. Finally, $\mu_1+\ldots+\mu_{N_1}+2(\nu_1+\ldots+\nu_2)=n$.

Proof. By the fundamental theorem of algebra we know that the polynomial $p(x)$ can be factorized as follows

$$p(x)=a_n(x-z_1)^{m_1}\cdots(x-z_N)^{m_N}, \tag{3.4.2}$$

where $N\leq n, m_1,\ldots,m_N$ are natural numbers and $z_1,\ldots z_N$ are distinct complex numbers (i.e. the complex roots of $p(x)$). If needed, let us rename the roots so that $z_1,\ldots,z_{N_1}\in\mathbb{R}$ and $z_{N_1+1},\ldots,z_N\in\mathbb{C}\backslash\mathbb{R}$, for some natural number $N_1\leq N$ (possibly equal to zero). We now observe the following fact:

**Proposition 3.4.12**

If $p(x)$ is a real polynomial, then a complex number $w$ verifies $p(w)=0$ if and only $p(\bar{w})=0$, namely, $w$ is a root of $p(x)$ if and only if $\bar{w}$ is a root of $p(x)$.

This Proposition is straightforward as soon as one utilizes the commutativity between the operations (i.e. sum and product) and the conjugation in $\mathbb{C}$. Indeed, for any root $w$ of $p(x)$, one has $p(w)=0$, so that

$$0=\overline{p(w)}=\overline{(a_nw^n+\ldots+a_1w_1+a_0)}=\bar{a}_n\bar{w}^n+\ldots+\bar{a}_1\bar{w}+\bar{a}_0=a_n\bar{w}^n+\ldots+a_1\bar{w}+a_0=p(\bar{w}).$$

□

Let us continue with the proof of the theorem. Set $x_k:=z_k, \mu_k=m_k, \forall k=1,\ldots,N_1$ and observe that, because of the previous Proposition, the number $N-N_1$ is even, i.e. $\exists N_2\in\mathbb{C}$ such that $N-N_1=2N_2$. Hence, by (3.4.2) one gets

$$p(x)=a_n(x-x_1)^{\mu_1}\cdot\dots\cdot(x-x_{N_1})^{\mu_{N_1}}\cdot(x-w_1)^{\nu_1}(x-\bar{w}_1)^{\nu_1}\cdot\dots\cdot(x-w_{N_2})^{\nu_{N_2}}(x-\bar{w}_{N_2})^{\nu_{N_2}}$$

where we have set $w_h:=z_{N_1+h}\ \nu_h:=m_{N_1+m_h}, \forall h=1,\ldots,N_2$. Now it is sufficient to observe that, $\forall h=1,\ldots,N_2$,

$$(x-w_h)(x-\bar{w}_h)=x^2+b_hx+c_h$$

where $b_h,c_h$ are (*real !* ) numbers defined as $b_h:=2\text{Re }w_h$ and $c_h:=w_h\bar{w}_h$, to get the thesis of the theorem. Let us remark that the strict inequality $\Delta_h=b_h^2-4c_h<0$ is satisfied, because

$$b_h^2-4a_hc_h=(2\text{Re }w_h)^2-4w_h\bar{w}=4\text{Re }w_h)^2-4(\text{Re }w_h)^2-4((\text{Re }w_h)^2)-4(\text{Im }w_h)^2=-4(\text{Im }w_h)^2<0$$

(the last inequality being strict, since $w_h\in\mathbb{C}\backslash\mathbb{R}$). □

From the previous theorem one deduce the the following fact:

Corollary 3.4.13. A real polynomial $p(x)$ of odd degree has at least one real root.

Proof. If by contradiction $p(x)$ had no real roots, it would be factorizable as a product of even-degree polynomials, so that its degree would be even as well, contrary to the hypothesis. □

Example 3.4.14. *Prove that $z_1=3, z_2=-4$ are roots of the polynomial*

$$P(z)=z^4+z^3-11z^2+z-12$$

*and then find all the roots of $P(z)$.*

Sol. — In view of the Fond.Thm of Alg. we know in advance that there must exist exactly 4 solutions, some of which may well be coinciding. The fact that $z_1 = 3, z_2 = -4$ are roots is simply obtained by substitution, for

$$P(3) = P(-4) = 0.$$

By Ruffini's Theorem we then get that $P$ has both $(z+4)$ and $(z-3)$ as factors.

Hence, we have

$$P(z) = (z-3)(z+4)Q(z), \qquad \forall z \in \mathbb{C}$$

for some degree-2 polynomial $Q(\cdot)$. To determine $Q(\cdot)$ we simply divide $P(z)$ by $(z-3)(z+4) = z^2 + z - 12$. Using the well-known method for polynomial division one finds that

$$Q(z) = \frac{P(z)}{z^2 + z - 12} = z^2 + 1$$

. So the other two roots (possibly coinciding) $z_3, z_4$ of $P(z)$ must be the roots of $z^2 + 1$, namely the solutions of $z^2 = -1$. Hence

$$z_3 = i, \;\; z_4 = -i$$

Summarizing the above findings, we have that following factorization holds true:

$$P(z) = (z^2+1)(z-3)(z+4) = (z+i)(z-i)(z-3)(z+4),$$

Observe that, as a real polynomial, $P$ can be factorized as follows:

$$P(x) = (x-3)(x+4)(x^2+1)$$

Example 3.4.15. *Solve*

$$9iz^5 = 16\bar{z}.$$

Sol. — Notice that $z = 0$ is a solution. Let $z$ be any solution and write it in polar notation $z = \rho e^{i\theta}$. Since $z^5 = \rho^5 e^{i5\theta}$, $\bar{z} = \rho e^{-i\theta}$ and $i = e^{i\frac{\pi}{2}}$, the equation becomes

$$9e^{i\frac{\pi}{2}}\rho^5 e^{i5\theta} = 16\rho e^{-i\theta},$$

that is

$$9e^{\frac{\pi}{2}}\rho^5 e^{i5\theta} = 16\rho e^{-i\theta}, \iff 9\rho^5 e^{\frac{\pi}{2}+5\theta} = 16\rho e^{-i\theta}.$$

This is an identity between two numbers in trigonometric form: we have

$$\begin{cases} 9\rho^5 = 16\rho, \\ \\ \dfrac{\pi}{2} + 5\theta = -\theta + k2\pi, \quad k \in \mathbb{Z}, \end{cases} \iff \begin{cases} \rho(9\rho^4 - 16) = 0, \\ \\ \theta = -\dfrac{\pi}{12} + k\dfrac{\pi}{3}, \qquad k \in \mathbb{Z}. \end{cases}$$

By the first we see that either $\rho = 0$ (solution $z = 0$ already discovered) or

$$\rho^4 = \frac{16}{9}, \iff \rho = \frac{2}{\sqrt{3}}. \quad \square$$

As for the arguments (up to multiples of $2\pi$), we get

$$\theta = \frac{-\pi}{12}, \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}.$$

To obtain the algebraic expressions of these roots let us use some trigonometry:

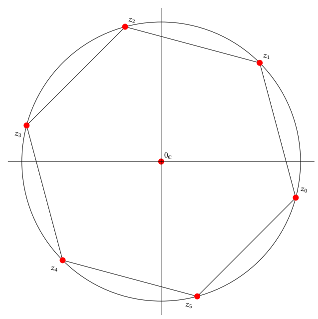
$$-\cos(11\pi/12) = \sin(7\pi/12) = \cos(-11\pi/12) = cos(-\pi/12) = cos(\pi/12) = \sqrt{\frac{1 + cos(\pi/6)}{2}} = \frac{1}{2}\sqrt{2+\sqrt{3}},$$

$$-\sin(11\pi/12) = \cos(7\pi/12) = \sin(-11\pi/12) = sin(-\pi/12) = -sin(\pi/12) = -\sqrt{\frac{1-2cos(\pi/6)}{2}} = -\frac{1}{2}\sqrt{2-\sqrt{3}}$$

With similar computation and exploiting symmetries, in the end we find the following algebraic expressions of the roots:

$$z_0 = 0, \quad z_1 = \frac{1}{\sqrt{3}}\left(\sqrt{2+\sqrt{3}} - i\sqrt{2-\sqrt{3}}\right), \quad z_2 = \frac{\sqrt{2}}{\sqrt{3}}(1+i), \quad z_3 = \frac{1}{\sqrt{3}}\left(-\sqrt{2-\sqrt{3}} + i\sqrt{2+\sqrt{3}}\right)$$

$$z_4 = \frac{1}{\sqrt{3}}\left(-\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right), \quad z_5 = \frac{\sqrt{2}}{\sqrt{3}}(-1-i), \quad z_6 = \frac{1}{\sqrt{3}}\left(\sqrt{2-\sqrt{3}} - i\sqrt{2+\sqrt{3}}\right)$$



Remark 3.4.16. By misinterpreting the equation as a polynomial equation, we might have conjectured that this equation has 5 distinct solutions, so that our result would have contradicted the Fundamental Theorem of Algebra. In fact, this is not a polynomial equation, which allows for 7 solutions.

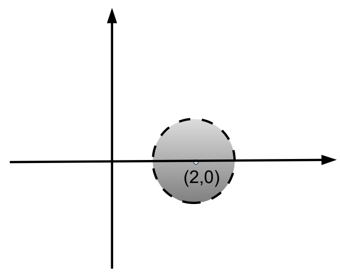
Example 3.4.17. *Plot the set* $S = \left\{z \in \mathbb{C} : \operatorname{Im} \frac{i(z-3)}{z-1} < 0\right\}$ *on the Gauss plane.*

Sol. — Writing $z = x + iy$ we get

$$\operatorname{Im} \frac{i(z-3)}{z-1} = \operatorname{Im} \frac{-y + i(x-3)}{(x-1)+iy} = \operatorname{Im} \frac{(-y+i(x-3))((x-1)-iy)}{((x-1)+iy)((x-1)-iy)} =$$

$$= \operatorname{Im}\left(\frac{-y(x-1) + (x-3)y + i((x-3)(x-1)+y^2)}{(x-1)^2+y^2}\right) = \frac{(x-3)(x-1)+y^2}{(x-1)^2+y^2} = \frac{x^2-4x+3+y^2}{(x-1)^2+y^2} < 0$$

if and only if $x^2 + y^2 - 4x + 3 < 0$.
Hence the plot set $S$ on the Gauss plane is represented by $\tilde{S} := \{(x, y) \ : x^2+y^2-4x+3 < 0.\}$ Since $x^2+y^2-4x+3$ is the equation of the circumference $C$ of center $(2, 0)$ and radius 1, the set $S$ is represented by the circle internal to the circumference $C$.




## 3.5. Exercises

EXERCISE 3.5.1. *Determine $a \in \mathbb{R}$ such that*

$$z = \frac{i}{a + 1 - ia}$$

*is imaginary.*

EXERCISE 3.5.2. *Solve and plot solutions (if any) in Gauss plane:*

*i)* $\bar{z}^2 + 2iz + 2\bar{z} = 0_{\mathbb{C}}$.
*ii)* $\bar{z}\mathrm{Im}\, z - z|z| = 0$.
*iii)* $iz(\bar{z} + 1) - |z|\mathrm{Re}\, z = 0$.
*iv)* $(\mathrm{Re}\, z)(\mathrm{Im}\, z^2) + z^2 + |z|^2 = 0$.

EXERCISE 3.5.3. *Plot following sets in $\mathbb{C}$:*

*i)* $\left\{z \in \mathbb{C} : \left|\frac{z-4}{z+4}\right| \geqslant 3\right\}$.
*ii)* $\left\{z \in \mathbb{C} : \mathrm{Im}\, \frac{i(z-3)}{z-1} < 0\right\}$.
*iii)* *(★)* $\left\{x \in \mathbb{C} : \mathrm{Re}\, \frac{z}{z+1} \leqslant \mathrm{Im}\, \frac{\bar{z}+1}{z+1}\right\}$.
*iv)* $\{z \in \mathbb{C} : \mathrm{Im}\, z - |z + \bar{z}|^2 > 1,\ \mathrm{Re}\, z > 0,\ \mathrm{Im}\, z < 2\}$.
*v)* $\left\{z \in \mathbb{C} : |z^2 - 1| < |\bar{z}^2 + 1|\right\}$.
*vi)* *(★)* $\left\{z \in \mathbb{C}\ :\ \mathrm{Im}\, \frac{z+1}{z-i} \geqslant 0,\ |z - 1 - i| \leq 1\right\}$.

Exercise 3.5.4. *Determine $z \in \mathbb{C}$ such that*

$$\begin{cases} z^2\bar{z} - \bar{z}z = -\bar{z}, \\ \\ (z^3 + \bar{z})^3 = 1. \end{cases}$$

Exercise 3.5.5. *Let $\alpha \in \mathbb{R}$ and*

$$S_\alpha := \{z \in \mathbb{C} : z\bar{z} + (-1+i)z + (-1-i)\bar{z} + \alpha < 0\}.$$

*Say for which $\alpha \in \mathbb{R}$ set $S_\alpha$ is non empty and draw it in the complex plane.*

Exercise 3.5.6. *Determine the value of $\alpha \in \mathbb{R}$ such that the system*

$$\begin{cases} \mathrm{Re}\ \bar{z}(z - 2i) = \alpha, \\ \\ \mathrm{Im}\ z = \mathrm{Re}\ z, \end{cases}$$

*admits precisely just one solution.*

Exercise 3.5.7 (★). *Solve*

$$|z|z^2 + \mathrm{Re}\ z\mathrm{Im}\ z - |z|^2 z = 0, \quad z \in \mathbb{C}.$$

Exercise 3.5.8. *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be defined as*

$$f(z) := -6i - |z|^2 i.$$

*Compute cubic roots of $f(i+1)$.*

Exercise 3.5.9. *Determine $\lambda \in \mathbb{C}$ in such a way that $z_0 = -i$ be a root for*

$$p(z) := z^8 + iz^7 + i2z^5 + \lambda z^4.$$

*For such $\lambda$, determine also all other roots of $p$.*

Exercise 3.5.10. *Solve:*

*i)* $z^8 = i\bar{z}|z|$.
*ii)* $z^4|z|^2 = z$.
*iii)* $z|z|^2 - i4\bar{z} = 0$.
*iv)* $8z^3|z| = \bar{z}$.
*v)* $\bar{z}^3|\bar{z}|^2 = 2z^4$.
*vi)* $z^2|i\bar{z}|^4 + i|z^2|\bar{z}^2 = 0$.

CHAPTER 4

# Sequences

## 4.1. Sequences

We start by formalizing the concept of sequence:

**Definition 4.1.1**

A **numerical sequence** (shortly, a sequence) is a function $a : \mathbb{N} \longrightarrow \mathbb{R}$. We will denote it by the symbol $(a_n) \subset \mathbb{R}$, where $a_n := a(n)$, $n \in \mathbb{N}$ is called **element** of the sequence, $n$ is called **index**.

An visual representation of a sequence is just given by the plot of the graph of the function $n \longmapsto a_n$.

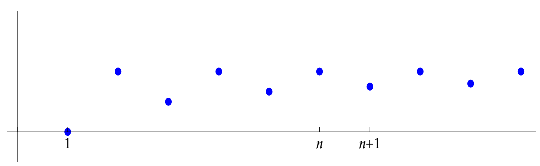



FIGURE 1. Graph of the sequence $a_n = \frac{n+(-1)^n}{n+1}$.

**4.1.1. Finite limit.** We want to make precise the idea $a_n \longrightarrow \ell \in \mathbb{R}$ as $n$ that becomes big. The idea is simple: the distance between $a_n$ and $\ell$ must become small as $n$ becomes big. Of course small and big are subjective concepts, we need a more formal definition:

**Definition 4.1.2**

Let $(a_n) \subset \mathbb{R}$. We say that $a_n \longrightarrow \ell \in \mathbb{R}$ (we read as: $(a_n)$ **tends to** $\ell$ **as** $n$ **tends to** $+\infty$) if

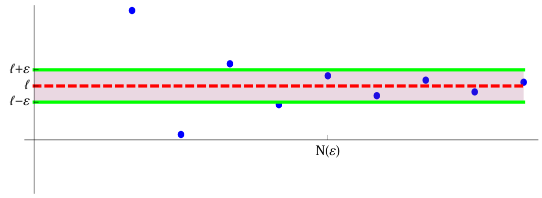
$$\forall \varepsilon > 0,\ \exists N \in \mathbb{R}\ :\ |a_n - \ell| \leqslant \varepsilon,\ \forall n \geqslant N. \tag{4.1.1}$$

We write also $\lim_{n\to+\infty} a_n := \ell$.

The (4.1.1) has a precise geometrical meaning. Writing it as

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{R}\ :\ \ell - \varepsilon \leqslant a_n \leqslant \ell + \varepsilon,\ \forall n \geqslant N,$$

we read: for any $\varepsilon > 0$ fixed (intuitively "small") we find an *initial index* (think to a time) $N$ such that, starting by the index $\geq N$ the point $(n, a_n)$ lies in the strip between quotes $\ell - \varepsilon$ and $\ell + \varepsilon$. As the picture suggests, as $\varepsilon$ gets smaller, $N$ gets bigger.




EXAMPLE 4.1.3. *Show that*
$$\frac{1}{n} \longrightarrow 0.$$

SOL. — It is clear that as $n$ gets bigger, $\frac{1}{n}$ gets smaller close to 0. To show that the (4.1.1) holds, we have to **fix** $\varepsilon > 0$ and **find** $N$ such that
$$\left|\frac{1}{n} - 0\right| \leqslant \varepsilon,\ \forall n \geqslant N.$$
Now,
$$\left|\frac{1}{n} - 0\right| \leqslant \varepsilon \iff \frac{1}{n} \leqslant \varepsilon \iff n \geqslant \frac{1}{\varepsilon} =: N.$$
□

EXAMPLE 4.1.4. *Show that*
$$\frac{n}{n+1} \longrightarrow 1.$$

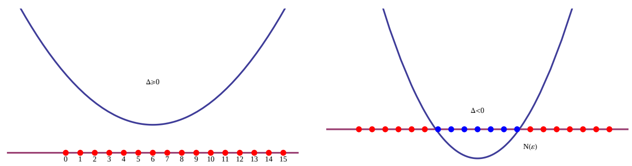
SOL. — Intuitively it is natural: as $n$ gets bigger, $n$ and $n+1$ are very "similar" numbers, so their ration should be approximatively 1. Let's check the (4.1.1), in this case translated as
$$\forall \varepsilon > 0,\ \exists N\ :\ \left|\frac{n}{n+1} - 1\right| \leqslant \varepsilon,\ \forall n \geqslant N.$$
Let's look to solutions of the inequality $\left|\frac{n}{n+1} - 1\right| \leqslant \varepsilon$. We have
$$\begin{aligned} \left|\tfrac{n}{n+1} - 1\right| \leqslant \varepsilon, \quad &\iff \quad -\varepsilon \leqslant \tfrac{n}{n+1} - 1 \leqslant \varepsilon, \quad \iff \quad 1 - \tfrac{n}{n+1} \leqslant \varepsilon, \quad \iff \quad (n+1) - n \leqslant \varepsilon(n+1), \\ &\iff \quad 1 \leqslant \varepsilon(n+1), \qquad\quad \iff \quad n \geqslant \tfrac{1}{\varepsilon} - 1. \end{aligned}$$
But then, if $N := \frac{1}{\varepsilon} - 1$ then (4.1.1) holds. □

EXAMPLE 4.1.5. *Show that*
$$\frac{n}{n^2+1} \longrightarrow 0.$$

SOL. — Let $\varepsilon > 0$. We have to find $N$ such that
$$\left|\frac{n}{n^2+1} - 0\right| \leqslant \varepsilon,\ \forall n \geqslant N.$$
Let's study the inequality $\left|\frac{n}{n^2+1} - 0\right| \leqslant \varepsilon$. We have
$$\left|\frac{n}{n^2+1} - 0\right| \leqslant \varepsilon, \iff \frac{n}{n^2+1} \leqslant \varepsilon, \iff \varepsilon(n^2+1) - n \geqslant 0, \iff \varepsilon n^2 - n + \varepsilon \geqslant 0.$$
This is a second degree inequality in $n$, (here $n \in \mathbb{N}$). Recalling the fundamental facts about second-order inequalities, being $\Delta = 1 - 4\varepsilon$ we have the following cases:

- $\Delta < 0$ (this happens iff $1 - 4\varepsilon \leqslant 0$, that is $\varepsilon \geqslant \frac{1}{4}$): the inequality is fulfilled for any $n \in \mathbb{N}$. In this case, we could take $N = 0$.

- $\Delta \geqslant 0$ (iff $0 < \varepsilon < \frac{1}{4}$): the solutions of the inequality are
$$n \leqslant \frac{1-\sqrt{1-4\varepsilon}}{2}, \quad n \geqslant \frac{1+\sqrt{1-4\varepsilon}}{2}.$$
The first one may be doesn't have integer solution, but the second has solutions all the integers bigger than $N = \frac{1+\sqrt{1-4\varepsilon}}{2}$




In any case (that is, for any $\varepsilon > 0$) we find $N$ such that any $n \geqslant N$ is a solution of the inequality. □

Remark 4.1.6. *Limit characterization may well be used for sequences $(a_n) \subset \mathbb{C}$ (where of course the limit $\ell \in \mathbb{C}$ and the modulus is the modulus of a complex number). Let us see an example: show that*
$$\frac{1}{n} + i\frac{n+1}{n} \longrightarrow i.$$

Sol. — Indeed, we have to prove that
$$\forall \varepsilon > 0, \ \exists N \in \mathbb{R}, \ : \ \left| \left( \frac{1}{n} + i\frac{n+1}{n} \right) - i \right| \leqslant \varepsilon, \ \forall n \geqslant N.$$
Notice that
$$\left| \left( \frac{1}{n} + i\frac{n+1}{n} \right) - i \right| = \left| \frac{1}{n} + i\frac{1}{n} \right| = \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{\sqrt{2}}{n} \leqslant \varepsilon,$$
iff $n \geqslant \frac{\sqrt{2}}{\varepsilon}$. Thus, taking $N = \frac{\sqrt{2}}{n}$ we get the conclusion. □

**4.1.2. Infinite limit.** As we have seen in the introduction, a second important situation is when $a_n$ gets big without any bound:

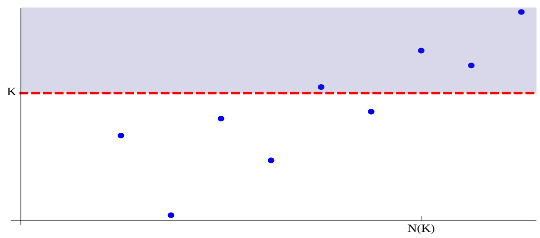
**Definition 4.1.7**

Let $(a_n) \subset \mathbb{R}$. We say that $a_n \longrightarrow +\infty$ if

(4.1.2) $$\forall K \in \mathbb{R}, \ \exists N \in \mathbb{R} \ : \ a_n \geqslant K, \ \forall n \geqslant N$$

We write also $\lim_{n\to+\infty} a_n := +\infty$.
Similarly is defined $\lim_{n\to+\infty} a_n = -\infty$:
$$\forall K \in \mathbb{R}, \ \exists N \in \mathbb{R} \ : \ a_n \leqslant K, \ \forall n \geqslant N.$$

Also in this case it is useful to have a picture of what does it means $a_n \longrightarrow +\infty$: the (4.1.2) says that for any $K$ fixed (intuitively "big and positive") we find an *initial time* $N$ such that from this on the point $(n, a_n)$ has quote $a_n \geqslant K$.




Example 4.1.8. *Show that*

$$\frac{n^2+1}{n+1} \longrightarrow +\infty.$$

Sol. — Fix $K > 0$: we have to find $N$ such that

$$\frac{n^2+1}{n+1} \geqslant K,\ \forall n \geqslant N.$$

Studying the inequality $\frac{n^2+1}{n+1} \geqslant K$ with $n \in \mathbb{N}$, we have

$$\frac{n^2+1}{n+1} \geqslant K, \overset{n\in\mathbb{N},\ n\geqslant 0}{\Longleftrightarrow} n^2+1 \geqslant K(n+1), \iff n^2 - Kn + (1-K) \geqslant 0,$$

that is a second degree inequality in $n$. Let $\Delta := K^2 - 4(1-K)$. We have

- if $\Delta < 0$ the inequality is fulfilled for every $n \in \mathbb{N}$: this means we can take $N = 0$.
- if $\Delta \geqslant 0$, the solutions are

$$n \leqslant \frac{K-\sqrt{\Delta}}{2},\quad n \geqslant \frac{K+\sqrt{\Delta}}{2}.$$

The first, has at most a finite number of solutions in natural numbers. From the second, setting $N := \frac{K+\sqrt{\Delta}}{2}$ we see that any $n \geqslant N$ is a solution.

In any case, we are able to find an initial index $N$. □

Example 4.1.9. *Show that*

$$\frac{n}{1-\sqrt{n}} \longrightarrow -\infty.$$

Sol. — We have to show that

$$\forall K < 0,\ \exists N,\ :\ \frac{n}{1-\sqrt{n}} \leqslant K,\ \forall n \geqslant N.$$

As $n \geqslant 2$,

$$\frac{n}{1-\sqrt{n}} \leqslant K, \iff n \geqslant (1-\sqrt{n})K, \iff n - K \geqslant -K\sqrt{n}.$$

Now: $n - K > 0$ (because $K < 0$) and also $-K\sqrt{n} > 0$. Squaring,

$$\frac{n}{1-\sqrt{n}} \leqslant K, \iff (n-K)^2 \geqslant K^2 n, \iff n^2 + 2Kn + K^2 - K^2 n \geqslant 0, \iff n^2 + (K^2+2K)n + K^2 \geqslant 0.$$

We meet again a second degree inequality. Setting $\Delta := (K^2+2K)^2 - 4K^2$ we have

- if $\Delta < 0$ every $n$ is solution, therefore being $n \geqslant 2$ we can take $N := 2$;

- if $\Delta \geqslant 0$ then the solutions for the inequality are

$$n \leqslant \frac{-(K^2+2K)-\sqrt{\Delta}}{2}, \quad n \geqslant \frac{-(K^2+2K)+\sqrt{\Delta}}{2}.$$

Now, taking $N := \frac{-(K^2+2K)+\sqrt{\Delta}}{2}$ we have that any $n \geqslant N, 2$ is a solution.

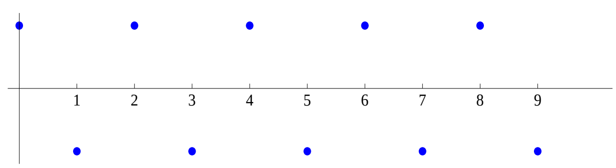
Therefore, we are always able to find $N$ such that $\frac{n}{1-\sqrt{n}} \leqslant K$ for any $n \geqslant N$. $\square$

Here's behavior of some basic quantities (checks are left for exercise):

$$n^\alpha \longrightarrow \begin{cases} +\infty, & \alpha > 0, \\ 1, & \alpha = 0, \\ 0, & \alpha < 0. \end{cases} \quad a^n \longrightarrow \begin{cases} +\infty, & a > 1, \\ 1, & a = 1, \\ 0, & 0 < a < 1. \end{cases} \quad \log_b n \longrightarrow \begin{cases} +\infty, & b > 1, \\ \\ -\infty, & 0 < b < 1. \end{cases}$$

**4.1.3. Non existence.** A sequence having a finite limit is shortly called *convergent*, while if the limit is infinite is called *divergent*. There is a third possible situation: a sequence might not have any limit (finite or less). For example, consider the sequence

$$a_n := (-1)^n, \quad \text{that is } +1, \ -1, \ +1, \ -1, \ \dots$$




It seems evident that such a sequence cannot have a limit. To prove this, we should prove that none of (4.1.1) and (4.1.2) are fulfilled. We take an alternative and more efficient path. We start by the

**Definition 4.1.10**

Let $(a_n) \subset \mathbb{R}$. Given a strictly increasing sequence of indexes

$$n_0 < n_1 < n_2 < \dots < n_k < n_{k+1} < \dots$$

The sequence

$$a_{n_0}, a_{n_1}, a_{n_2}, \dots, a_{n_k}, a_{n_{k+1}}, \dots \equiv: (a_{n_k})$$

is called **subsequence** of $(a_n)$. Notation: $(a_{n_k}) \subset (a_n)$.

EXAMPLE 4.1.11. *Let $(a_n) \subset \mathbb{R}$. Then the following are subsequences of $(a_n)$:*

- $(a_0, a_2, a_4, \dots, a_{2k}, \dots) \equiv (a_{2k})$;
- $(a_1, a_3, a_5, \dots, a_{2k+1}, \dots) \equiv (a_{2k+1})$;
- $(a_0, a_3, a_6, a_9, a_{12}, \dots, a_{3k}, \dots) \equiv (a_{3k})$;
- $(a_0, a_1, a_4, a_9, a_{16}, a_{25}, \dots, a_{k^2}, \dots) \equiv (a_{k^2})$;
- $(a_1, a_2, a_4, a_8, a_{16}, a_{32}, \dots, a_{2^k}, \dots) \equiv (a_{2^k})$;
- $(a_1, a_2, a_3, a_5, a_7, a_{11}, \dots) \equiv (a_{prime})$.

*However, for example,* $(a_1, a_0, a_3, a_2, \dots) = (a_{k+(-1)^k})$ *is not a subsequence of* $(a_n)$*. In this case,* $n_k = k + (-1)^k$ *is not an increasing sequence of indexes.* $\square$

It seems intuitively clear that *if a sequence has a limit, then all its subsequences have the same limit*. This is the content of the

**Proposition 4.1.12**

Let $(a_n) \subset \mathbb{R}$. If

$$a_n \longrightarrow \ell \in \mathbb{R} \cup \{\pm\infty\}, \implies a_{n_k} \longrightarrow \ell, \ \forall (a_{n_k}) \subset (a_n).$$

In particular: if $(a_{n_k}), (a_{m_k}) \subset (a_n)$ are such that

$$a_{n_k} \longrightarrow \ell_1, \ a_{m_k} \longrightarrow \ell_2, \ \text{with } \ell_1 \neq \ell_2,$$

then $(a_n)$ cannot have a limit.

Proof. Consider the case $\ell \in \mathbb{R}$ (leaving $\ell = \pm\infty$ to the reader as exercise). We have:

$$a_n \longrightarrow \ell, \iff \forall \varepsilon > 0, \exists N, : |a_n - \ell| \leqslant \varepsilon, \ \forall n \geqslant N.$$

Now, being $n_k \nearrow$ and $n_k \in \mathbb{N}$, it is clear that $n_k \geqslant k$. Therefore

$$|a_{n_k} - \ell| \leqslant \varepsilon, \ \forall k \geqslant N. \quad \square$$

Example 4.1.13. $(-1)^n$ *does not have any limit.*

Sol. — Indeed: $a_{2k} = (-1)^{2k} \equiv 1 \longrightarrow 1$ while $a_{2k+1} = (-1)^{2k+1} \equiv -1 \longrightarrow -1$. $\square$

We quote two other sequences without limit, even if it is very hard to prove that this: $(\sin n)$, $(\cos n)$.

### 4.2. Limits: main properties

In this section, we collect the main properties of the limit of sequences. We start with a question: is it possible that a sequence would have two limits?

**Proposition 4.2.1: Uniqueness**

If $\lim_n a_n$ exists, it is unique.

Proof. Suppose that $a_n \longrightarrow \ell_1$ and $a_n \longrightarrow \ell_2$ with $\ell_1, \ell_2 \in \mathbb{R} \cup \{\pm\infty\}$ and $\ell_1 \neq \ell_2$.

**Case $\ell_1, \ell_2 \in \mathbb{R}$.** The idea is simple: because $a_n \longrightarrow \ell_1$, sooner or later $a_n$ will be so close to $\ell_1$ that it cannot be close to $\ell_2$. Precisely: let $d := |\ell_1 - \ell_2|$ be the distance between $\ell_1$ and $\ell_2$ and let's take $\varepsilon := \frac{d}{4}$. By definition,

$$\exists N_1, : |a_n - \ell_1| \leqslant \varepsilon, \ \forall n \geqslant N_1, \ \text{ and } \exists N_2, : |a_n - \ell_2| \leqslant \varepsilon, \ \forall n \geqslant N_2.$$

But then, if $N := \max\{N_1, N_2\}$ the two previous properties hold for any $n \geqslant N$ and

$$|\ell_1 - \ell_2| \leqslant |\ell_1 - a_n| + |a_n - \ell_2| \leqslant \varepsilon + \varepsilon = 2\varepsilon = 2\frac{d}{4} = \frac{d}{2} < d = |\ell_1 - \ell_2|,$$

which is clearly impossible. It follows that $\ell_1 = \ell_2$.

**Cases $\ell_1 \in \mathbb{R}$, $\ell_2 = \pm\infty$ and $\ell_1 = -\infty$, $\ell_2 = +\infty$:** exercise. □

In short, a sequence has the same sign of its limit. Here's a precise statement:

**Proposition 4.2.2: Permanence of sign**

Assume $a_n \longrightarrow \ell \in \mathbb{R} \cup \{\pm\infty\}$. Then

i) If $\ell > 0$ (included $\ell = +\infty$) exists $N \in \mathbb{R}$ such that $a_n > 0$ for any $n \geqslant N$.

ii) If exists $N \in \mathbb{R}$ such that $a_n \geqslant 0$ for any $n \geqslant N$ then $\ell \geqslant 0$.

Proof. Let us prove i). Suppose $\ell \in \mathbb{R}$ and $\ell > 0$ (the case $\ell = +\infty$ is similar and is left as exercise). Let us take $\varepsilon := \frac{\ell}{2}$ into (4.1.1): there exists then $N =: N$ such that

$$\ell - \varepsilon \leqslant a_n \leqslant \ell + \varepsilon,\ \forall n \geqslant N,\ \Longrightarrow\ a_n \geqslant \ell - \varepsilon = \ell - \frac{\ell}{2} = \frac{\ell}{2} > 0,\ \forall n \geqslant N.$$

Let us prove ii) . Suppose, by contradiction, that $\ell < 0$. By the first statement it follows that there exists $\widetilde{N}$ such that $a_n < 0$ for every $n \geqslant \widetilde{N}$. However, this is a contradiction because if $n \geqslant \max\{N, \widetilde{N}\}$ we would have $a_n < 0 < a_n$. □

Remark 4.2.3. Unfortunately, we **cannot** say that, if $a_n \longrightarrow \ell$ then

- $\ell > 0$ iff $a_n > 0$ for $n \geqslant N$: indeed $\Longrightarrow$ is true (this is statement i) of the above Proposition) *but* $\Longleftarrow$ *might be false*, as in the case $a_n = \frac{1}{n} > 0$ and $a_n \longrightarrow 0$;
- $\ell \geqslant 0$ iff $a_n \geqslant 0$ for $n \geqslant N$:$\Longleftarrow$ is true (this is statement ii) of the above Proposition) but $\Longrightarrow$ *might be false*, as in the case $a_n = \frac{(-1)^n}{n} \longrightarrow 0$ but $a_n$ takes $\pm$ sign infinitely many times.

The previous Thm emphasizes the following important concept: given a sequence $(a_n) \subset \mathbb{R}$ we say that a certain property $p(a_n)$ is **definitely true** if

$$\exists N \in \mathbb{R},\ :\ p(a_n),\ \forall n \geqslant N.$$

In this case we will write shortly $p(a_n)$ *definitely*. For instance, *permanence of sign* may be restated in the following form: if $a_n \longrightarrow \ell$ *then*

- *if $\ell > 0$ then $a_n > 0$ definitely;*
- *if $a_n \geqslant 0$ definitely, then $\ell \geqslant 0$.*

If you are between two policemen, you do not have any alternative than following them. This is the sense of the

**Theorem 4.2.4: two policemen theorem**

Let $(a_n), (b_n), (c_n) \subset \mathbb{R}$ such that

i) $a_n \leqslant b_n \leqslant c_n$ definitely;

ii) $a_n \longrightarrow \ell$, $c_n \longrightarrow \ell$, $\ell \in \mathbb{R} \cup \{\pm\infty\}$.

Then $b_n \longrightarrow \ell$.

Proof. We consider only the case $\ell \in \mathbb{R}$, the others are left as exercise. We have to prove that

$$\forall \varepsilon > 0,\ \exists N\ :\ |b_n - \ell| \leqslant \varepsilon,\ \forall n \geqslant N,\ \iff\ \ell - \varepsilon \leqslant b_n \leqslant \ell + \varepsilon,\ \forall n \geqslant N.$$

By assumption

$$a_n \longrightarrow \ell,\ \implies\ \exists N_1(\varepsilon)\ :\ \ell - \varepsilon \leqslant a_n \leqslant \ell + \varepsilon,\ \forall n \geqslant N_1(\varepsilon),$$

$$c_n \longrightarrow \ell,\ \implies\ \exists N_2(\varepsilon)\ :\ \ell - \varepsilon \leqslant c_n \leqslant \ell + \varepsilon,\ \forall n \geqslant N_2(\varepsilon),$$

Moreover, by i) there exists $N$ such that $a_n \leqslant b_n \leqslant c_n$ for any $n \geqslant N$. Then, if $N := \max\{N_1, N_2, N\}$ the previous properties hold for any $n \geqslant N$. We deduce that

$$\ell - \varepsilon \leqslant a_n \leqslant b_n \leqslant c_n \leqslant \ell + \varepsilon,\ \forall n \geqslant N. \quad \square$$

Example 4.2.5. *Show that*

$$\frac{(-1)^n}{n} \longrightarrow 0.$$

Sol. — Indeed, being $-1 \leqslant (-1)^n \leqslant 1$ for any $n \in \mathbb{N}$, we have

$$-\frac{1}{n} \leqslant \frac{(-1)^n}{n} \leqslant \frac{1}{n},\ \forall n \geqslant 1.$$

Now: $-\frac{1}{n}$ and $\frac{1}{n}$ are the two policemen going to 0. Therefore, $\frac{(-1)^n}{n} \longrightarrow 0$. $\square$

This example suggests a general rule. Let us first introduce the following

**Definition 4.2.6**

We say that $(a_n) \subset \mathbb{R}$ is

- **bounded** if there exists $M$ such that $|a_n| \leqslant M$ for any $n \in \mathbb{N}$.
- **infinitesimal** if $a_n \longrightarrow 0$.

We have,

**Proposition 4.2.7**

If $(a_n)$ is bounded and $(b_n)$ is infinitesimal, then $(a_n b_n)$ is infinitesimal.

Example 4.2.8. *Compute*

$$\lim_{n \to +\infty} \frac{\sin n}{n}.$$

Sol. — Writing $\frac{\sin n}{n} = (\sin n) \cdot \frac{1}{n}$ and because $\sin n$ is bounded (being $|\sin n| \leqslant 1$ for any $n \in \mathbb{N}$) and $\frac{1}{n}$ null, their product is null: therefore the limit is 0. $\square$

### 4.3. The Bolzano–Weierstrass theorem

In this subsection, we put a first important brick on the future development of a very important *soft Analysis* tool: the Bolzano–Weierstrass Thm. In short, it says that any bounded sequence has at least a convergent subsequence. This is evident in the case of the sequence $(-1)^n$ but it is in general not evident (think to the sequence $\sin n$).

**Proposition 4.3.1**

Any convergent sequence is bounded.

Proof. Exercise. □

**Theorem 4.3.2: Bolzano–Weierstrass**

Any bounded sequence has a convergent subsequence.

Proof. The idea is easy: consider the "trace" left by the sequence on $\mathbb{R}$, that is the set

$$S := \{a_n \ : \ n \in \mathbb{N}\}.$$

For instance: if $a_n \equiv \xi$ (constant sequence), $S = \{\xi\}$; if $a_n = (-1)^n$ then $S = \{-1, +1\}$. And so on. Because $(a_n)$ is bounded, by assumption, we can say that

$$\exists \alpha_0, \beta_0 \in \mathbb{R} \ : \ \alpha_0 \leqslant a_n \leqslant \beta_0, \ \forall n \in \mathbb{N}, \implies S \subset [\alpha_0, \beta_0].$$

Now the point is the following:

- If $S$ is **finite**: it is clear that at least one of its elements has to be equal to $a_n$ for infinitely many $n$ (otherwise, if any $s \in S$ would be equal to $a_n$ only for a finite number of $n$, the set $S$ should be infinite). In this case, we have that there exists $(a_{n_k}) \subset (a_n)$ with $a_{n_k} \equiv s$ hence $a_{n_k} \longrightarrow s$, so we have the thesis.
- If $S$ is **infinite**, we construct the subsequence in the following way. First: divide $[\alpha_0, \beta_0]$ in two parts. At least one of them must contain infinite elements of $S$ (otherwise $S$ would be finite). Let's call

  $$[\alpha_1, \beta_1] \subset [\alpha_0, \beta_0], \text{ the half part such that } [\alpha_1, \beta_1] \cap S =: S_1 \text{ is infinite}$$

  Call $n_1$ such that $a_{n_1} \in [\alpha_1, \beta_1] \cap S \equiv S_1$. Now, repeat the argument with $S_1$: divide $[\alpha_1, \beta_1]$ in two parts: because $S_1$ is infinite, at least one of the two half must contain infinitely many points of $S_1$. Let's call

  $$[\alpha_2, \beta_2] \subset [\alpha_1, \beta_1], \text{ the half part such that } [\alpha_2, \beta_2] \cap S_1 =: S_2 \text{ is infinite.}$$

  It is clear that we can find $a_{n_2} \in [\alpha_2, \beta_2] \cap S_1 =: S_2$ with $n_2 > n_1$. Iterating this procedure we get:
  - intervals $[\alpha_k, \beta_k]$, each one being one half of the previous $[\alpha_{k-1}, \beta_{k-1}]$;
  - elements $a_{n_k} \in [\alpha_k, \beta_k]$ with $n_k > n_{k-1}$.

Then $(a_{n_k}) \subset (a_n)$. We say that $(a_{n_k})$ converges. To this aim, notice that, by construction

$$\alpha_k \leqslant a_{n_k} \leqslant \beta_k.$$

Moreover $\alpha_k \nearrow$ while $\beta_k \searrow$: being monotone sequences $\alpha_k \longrightarrow \alpha = \sup_k \alpha_k$ and $\beta_k \longrightarrow \beta = \inf_k \beta_k$ and because

$$0 \leqslant \beta - \alpha \leqslant \beta_k - \alpha_k \leqslant \frac{\beta_0 - \alpha_0}{2^k} \longrightarrow 0, \implies \alpha = \beta.$$

But then, by the two-policemen theorem it follows that $a_{n_k} \longrightarrow \alpha$. □

## 4.4. Rules of calculus

We need some rules of calculus to compute the limits in an efficient way. Fortunately, there are very simple rules that make most of the situations easy to handle. There are, however, a few number of exceptions where *no rules are available*: these cases are called *indeterminate forms*.

### 4.4.1. Finite Limits.

We start with the simple

**Proposition 4.4.1**

Let

$$a_n \longrightarrow \ell_1 \in \mathbb{R}, \quad b_n \longrightarrow \ell_2 \in \mathbb{R}.$$

Then

i) $a_n \pm b_n \longrightarrow \ell_1 \pm \ell_2$.
ii) $a_n \cdot b_n \longrightarrow \ell_1 \cdot \ell_2$.
iii) if $\ell_2 \neq 0$, $\frac{a_n}{b_n} \longrightarrow \frac{\ell_1}{\ell_2}$.

Proof. Let's prove only i). We have to prove that

$$\forall \varepsilon > 0, \exists N \ : \ |(a_n + b_n) - (\ell_1 + \ell_2)| \leqslant \varepsilon, \ \forall n \geqslant N.$$

Now

$$|(a_n + b_n) - (\ell_1 + \ell_2)| = |(a_n - \ell_1) + (b_n - \ell_2)| \leqslant |a_n - \ell_1| + |b_n - \ell_2|.$$

Fixed $\varepsilon > 0$ we have

$$a_n \longrightarrow \ell_1, \implies \exists N_1\left(\tfrac{\varepsilon}{2}\right) \ : \ |a_n - \ell_1| \leqslant \tfrac{\varepsilon}{2}, \ \forall n \geqslant N_1\left(\tfrac{\varepsilon}{2}\right),$$

$$b_n \longrightarrow \ell_2, \implies \exists N_2\left(\tfrac{\varepsilon}{2}\right) \ : \ |b_n - \ell_2| \leqslant \tfrac{\varepsilon}{2}, \ \forall n \geqslant N_2\left(\tfrac{\varepsilon}{2}\right).$$

Therefore, if

$$N := \max\left\{N_1\left(\frac{\varepsilon}{2}\right), N_2\left(\frac{\varepsilon}{2}\right)\right\}, \implies |a_n - \ell_1| \leqslant \frac{\varepsilon}{2}, \ |b_n - \ell_2| \leqslant \frac{\varepsilon}{2}, \ \forall n \geqslant N,$$

so

$$|(a_n + b_n) - (\ell_1 + \ell_2)| \leqslant |a_n - \ell_1| + |b_n - \ell_2| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ \ \forall n \geqslant N. \quad \square$$

Example 4.4.2. *Compute*

$$\lim_{n\to+\infty} \frac{\sqrt{n}-1}{\sqrt{n}+1}.$$

Sol. — We have

$$\frac{\sqrt{n}-1}{\sqrt{n}+1} = \frac{\sqrt{n}\left(1-\frac{1}{\sqrt{n}}\right)}{\sqrt{n}\left(1+\frac{1}{\sqrt{n}}\right)} = \frac{1-\frac{1}{\sqrt{n}}}{1+\frac{1}{\sqrt{n}}}.$$

Now, clearly

$$1-\frac{1}{\sqrt{n}} \longrightarrow 1,\ \ 1+\frac{1}{\sqrt{n}} \longrightarrow 1,\ \Longrightarrow\ \frac{1-\frac{1}{\sqrt{n}}}{1+\frac{1}{\sqrt{n}}} \longrightarrow \frac{1}{1} = 1. \quad \square$$

The case $\frac{a_n}{b_n}$ with $b_n \longrightarrow 0$ is not included in the previous list. In certain cases, we may have a rule. To show this, let us introduce an important

**Definition 4.4.3**

We say $a_n \longrightarrow \ell+$ if

i) $a_n \longrightarrow \ell$;
ii) $a_n > \ell$ definitely (that is for $n \geqslant N$).

We have

**Proposition 4.4.4**

Let

$$a_n \longrightarrow \ell \in \mathbb{R}\backslash\{0\},\ b_n \longrightarrow 0+,\ \Longrightarrow\ \frac{a_n}{b_n} \longrightarrow (\mathrm{sgn}\ell)\infty.$$

If the sign of the denominator is not definitely constant, the limit will not exists.

Example 4.4.5. *The limit of* $\frac{1}{\frac{(-1)^n}{n}} \equiv (-1)^n n$ *does not exist.* $\square$

The case $\frac{0}{0}$ leads to an *indeterminate form*. By this we mean a situation in which we cannot decide a priori what happens to the limit. This situation can be shown by a few examples that show that everything might happen:

- take $\frac{1/n^2}{1/n} = \frac{n}{n^2} = \frac{1}{n} \longrightarrow 0$;
- take $\frac{1/n}{1/n^2} = \frac{n^2}{n} = n \longrightarrow +\infty$;
- take $\frac{1/n}{1/n} \equiv 1 \longrightarrow 1$;
- take $\frac{(-1)^n/n}{1/n} \equiv (-1)^n$ and limit does not exist.

Thus, when we meet a $\frac{0}{0}$ everything may happen: a finite or infinite limit, or even the limit does not exist. This does not mean at all that *we cannot determine the limit*! Part of the techniques developed along this course are devoted to solve this kind of problems.

**4.4.2. Infinite and infinitesimal limits.** To treat with infinite limits is much more delicate. This because $\pm\infty$ are not numbers like any other real number, thus we need a bit of cautiousness. However, a number of properties are easy and natural:

**Proposition 4.4.6**

Suppose
$$a_n \longrightarrow \ell_1, \quad b_n \longrightarrow \ell_2.$$
Then

i) if $\ell_1 = \pm\infty$ and $\ell_2 \in \mathbb{R}$ then $a_n + b_n \longrightarrow \pm\infty$ (same sign of $\ell_1$).
ii) if $\ell_1 = \ell_2 = \pm\infty$ (same sign) then $a_n + b_n \longrightarrow \pm\infty$ (with the common sign of $\ell_1$ and $\ell_2$).

We use the following notation to summarize the content of the previous proposition:
$$(\pm\infty) + \ell = \pm\infty, \ (\ell \in \mathbb{R}), \quad (+\infty) + (+\infty) = +\infty, \quad (-\infty) + (-\infty) = -\infty.$$

Remark 4.4.7. *In the case $\ell_1 = +\infty$ and $\ell_2 = -\infty$ nothing can be said in principle. We can find examples that show very different behavior that cannot be described by an a priori rule. For this reason, we say that $(+\infty) + (-\infty)$ (or $(+\infty) - (+\infty)$) is an* **indeterminate form**. To understand better, let us see some examples:

- $a_n = n^2 \longrightarrow +\infty$, $b_n = -n \longrightarrow -\infty$. Then $a_n + b_n = n^2 - n$. It is however easy to show that $n^2 - n \longrightarrow +\infty$. For instance: $n^2 \geqslant 2n$ as $n \geqslant 2$, so
$$n^2 - n \geqslant 2n - n = n \longrightarrow +\infty.$$
- $a_n = n \longrightarrow +\infty$, $b_n = -n^2 \longrightarrow -\infty$. Here
$$a_n + b_n = n - n^2 \longrightarrow -\infty.$$
- $a_n = n + 1 \longrightarrow +\infty$, $b_n = -n \longrightarrow -\infty$. Here
$$a_n + b_n = (n + 1) - n = 1 \longrightarrow 1.$$
- $a_n = n + (-1)^n \geqslant n - 1 \longrightarrow +\infty$, $b_n = -n \longrightarrow -\infty$. Here
$$a_n + b_n = (n + (-1)^n) - n = (-1)^n,$$
and the limit does not exists. □

**Proposition 4.4.8**

Suppose
$$a_n \longrightarrow \ell_1, \quad b_n \longrightarrow \ell_2.$$
Then

i) if $\ell_1 = \pm\infty$ and $\ell_2 \in \mathbb{R}\backslash\{0\}$ then $a_n b_n \longrightarrow \mathrm{sgn}(\ell) \pm \infty$.
ii) if $\ell_1, \ell_2 \in \{\pm\infty\}$ then $a_n b_n \longrightarrow \mathrm{sgn}(\ell_1)\mathrm{sgn}(\ell_2)\infty$.

Like for the sum, we use the shortened notations

$$(+\infty)\cdot\ell = \mathrm{sgn}(\ell)\infty,\ (\ell\neq 0),\quad (-\infty)\cdot\ell = -\mathrm{sgn}(\ell)\infty,\ (\ell\neq 0),$$

$$(+\infty)\cdot(+\infty) = +\infty,\quad (+\infty)\cdot(-\infty) = -\infty,\quad (-\infty)\cdot(-\infty) = +\infty.$$

Remark 4.4.9. *The indeterminate form for the product is* $\pm\infty\cdot 0$*.* Here's some examples:

$$a_n = n \longrightarrow +\infty,\ \ b_n = \tfrac{1}{n} \longrightarrow 0,\qquad a_n b_n = n\tfrac{1}{n} = 1 \longrightarrow 1.$$

$$a_n = n^2 \longrightarrow +\infty,\ \ b_n = \tfrac{1}{n} \longrightarrow 0,\qquad a_n b_n = n^2\tfrac{1}{n} = n \longrightarrow +\infty,$$

$$a_n = n \longrightarrow +\infty,\ \ b_n = \tfrac{(-1)^n}{n} \longrightarrow 0,\quad a_n b_n = n\tfrac{(-1)^n}{n} = (-1)^n,\ \text{doesn't exist.}\quad \square$$

**Proposition 4.4.10**

Suppose

$$a_n \longrightarrow \ell_1,\quad b_n \longrightarrow \ell_2.$$

Then

i) if $\ell_1 \in \mathbb{R}$ and $\ell_2 \in \{\pm\infty\}$ then $\frac{a_n}{b_n} \longrightarrow 0$.
ii) if $\ell_1 \in \{\pm\infty\}$ and $\ell_2 \in \mathbb{R}\backslash\{0\}$ then $\frac{a_n}{b_n} \longrightarrow \mathrm{sgn}(\ell_1)\mathrm{sgn}(\ell_2)\infty$.

The mnemonic forms are

$$\frac{\ell}{\pm\infty} = 0,\ (\ell\in\mathbb{R}),\quad \frac{+\infty}{\ell} = \mathrm{sgn}(\ell)\infty,\ (\ell\neq 0),\quad \frac{-\infty}{\ell} = -\mathrm{sgn}(\ell)\infty,\ (\ell\neq 0).$$

To finish, we pass to the case of ration. Easily we have the rules

$$\frac{+\infty}{0+} = \frac{-\infty}{0-} = +\infty,\quad \frac{+\infty}{0-} = \frac{-\infty}{0+} = -\infty.$$

However, as for $\frac{0}{0}$, $\frac{\infty}{\infty}$ is an indeterminate form. Summarizing: the following are indeterminate forms:

$$(\pm\infty)+(\mp\infty),\ (\text{opposite signs}),\quad (\pm\infty)\cdot 0,\quad \frac{0}{0},\ \frac{\pm\infty}{\pm\infty}.$$

Example 4.4.11. *Compute*

$$\lim_{n\to+\infty}\frac{n^3-n^2+2}{n^3-n-5}$$

Sol. — At first sight, there are several indeterminate forms. In the numerator, we have $(+\infty)-(+\infty)+2 = (+\infty)-(+\infty)$. It is however easy to eliminate the problem: clearly, as $n$ is big, $n^3$ should be much bigger than $n^2$, driving the numerator to $+\infty$. For a precise argument, notice that writing

$$n^3-n^2+2 = n^3\left(1-\frac{1}{n}+\frac{2}{n^3}\right) \overset{(+\infty)\cdot(1-0+0)=(+\infty)\cdot 1=+\infty}{\longrightarrow} +\infty.$$

Notice that we used the rule $(+\infty)\cdot 1 = +\infty$. Similarly, at the denominator $n^3 - n - 5 = n^3\left(1 - \frac{1}{n^2} - \frac{5}{n^3}\right) \stackrel{(+\infty)\cdot 1}{\longrightarrow} +\infty$. Done this we meet another indeterminate form: the fraction appears as $\frac{+\infty}{+\infty}$. However, using previous factorizations,

$$\frac{n^3 - n^2 + 2}{n^3 - n - 5} = \frac{n^3\left(1 - \frac{1}{n} + \frac{2}{n^3}\right)}{n^3\left(1 - \frac{1}{n^2} - \frac{5}{n^3}\right)} = \frac{1 - \frac{1}{n} + \frac{2}{n^3}}{1 - \frac{1}{n^2} - \frac{5}{n^3}} \longrightarrow 1. \quad \square$$

Example 4.4.12. *Compute*

$$\lim_{n\to+\infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!}.$$

Sol. — Of course $n! \longrightarrow +\infty$ and similarly $(n+2)!$, $(n+1)! \longrightarrow +\infty$. Therefore, $(n+2)! + (n+1)! \longrightarrow +\infty$ while $(n+2)! - (n+1)!$ is an indeterminate form $(+\infty) - (+\infty)$. However, it seems evident that $(n+2)!$ is much bigger than $(n+1)!$ and noticed that $(n+2)! = (n+2)(n+1)!$, we have

$$(n+2)! - (n+1)! = (n+2)!\left(1 - \frac{(n+1)!}{(n+2)!}\right) = (n+2)!\left(1 - \frac{1}{n+2}\right) \stackrel{(+\infty)\cdot 1}{\longrightarrow} +\infty.$$

Now we have a form $\frac{+\infty}{+\infty}$, but using the previous trick,

$$\frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!} = \frac{1 + \frac{1}{n+2}}{1 - \frac{1}{n+2}} \longrightarrow 1. \quad \square$$

In these first examples emerges an idea: *"biggest" terms dominate*. For instance, for the difference $a_n - b_n$, we have

$$a_n - b_n = a_n\left(1 - \frac{b_n}{a_n}\right).$$

Now: $a_n$ is "bigger" than $b_n$ if $\frac{b_n}{a_n}$ is small! This is a crucial idea:

**Definition 4.4.13**

Given $(a_n), (b_n) \subset \mathbb{R}$ two infinite sequences, we say that

- $a_n$ **has higher order than** $b_n$ (notation $a_n \gg_{+\infty} b_n$) if
$$\frac{b_n}{a_n} \longrightarrow 0.$$
- $a_n$ **has the same order of** $b_n$ (notation $a_n \asymp b_n$) if
$$\frac{b_n}{a_n} \longrightarrow C \neq 0.$$

  In particular, if $C = 1$ we say that $b_n$ is **asymptotic** to $a_n$ (notation: $a_n \sim b_n$).

Therefore: if $a_n \gg_{+\infty} b_n$)

$$a_n - b_n = a_n\left(1 - \frac{b_n}{a_n}\right),$$

is no more an indeterminate form. However, if $a_n \sim_{+\infty} b_n$ the previous transformation change the indeterminate form $(+\infty) - (+\infty)$ into the form $(+\infty)\cdot 0$, so it is useless.

EXAMPLE 4.4.14. *Compute*

$$\lim_{n\to+\infty}\left(\sqrt{n+1}-\sqrt{n}\right).$$

SOL. — Clearly $\sqrt{n+1}, \sqrt{n} \longrightarrow +\infty$ so we have the indeterminate form $(+\infty)-(+\infty)$. Notice that even if $\sqrt{n+1} > \sqrt{n}$ is not true that $\sqrt{n+1} \gg \sqrt{n}$. Indeed

$$\frac{\sqrt{n}}{\sqrt{n+1}} = \sqrt{\frac{n}{n+1}} = \sqrt{1-\frac{1}{n+1}}.$$

It seems evident that being $1-\frac{1}{n+1} \longrightarrow 1$ then $\sqrt{1-\frac{1}{n+1}} \longrightarrow \sqrt{1} = 1$. This is actually a property of the root and of lots of functions called *continuity*. For the moment we do not care about it and we assume it is true. Therefore

$$\frac{\sqrt{n}}{\sqrt{n+1}} \longrightarrow 1, \iff \sqrt{n} \sim \sqrt{n+1}.$$

So none of the two terms is bigger and the factorization is useless. Then? An algebraic trick allows to proceed:

$$\sqrt{n+1}-\sqrt{n} = \left(\sqrt{n+1}-\sqrt{n}\right)\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \xrightarrow{\frac{1}{+\infty}} 0. \qquad \square$$

We have a similar classification for infinitesimals:

**Definition 4.4.15**

Given $(a_n), (b_n) \subset \mathbb{R}$ two infinitesimals (namely $a_n \to 0$, $b_n \to 0$), and assume that $b_n \neq 0$, definitely. We say that

- the infinitesimal $a_n$ **has higher order than** the infinitesimal $b_n$ if
$$\frac{a_n}{b_n} \longrightarrow 0.$$
- the infinitesimal $a_n$ **has the same order of** the infinitesimal $b_n$ if
$$\frac{a_n}{b_n} \longrightarrow C \neq 0.$$
In particular, if $C = 1$ we say that $b_n$ is **asymptotic** to $a_n$ (notation: $a_n \sim b_n$).

For instance, if $\alpha, \beta > 0$, the infinitesimal sequence $\frac{1}{n^\alpha}$, is of higher order than [resp. asymptotic to] the infinitesimal sequence $\frac{1}{(n+1)^\beta}$ if and only if $\alpha > \beta$ [resp. $\alpha = \beta$]

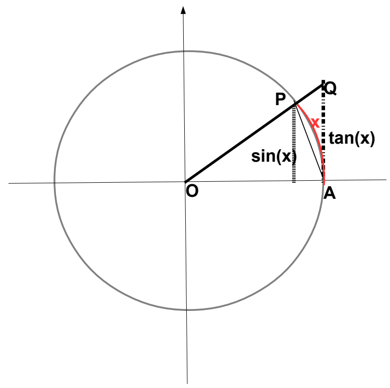
EXAMPLE 4.4.16. *Prove that the sequence* $\sin\left(\frac{1}{n}\right)$ *is (infinitesimal and) asymptotic to the (infinitesimal) sequence* $\frac{1}{n}$, *i.e.,*

$$\lim_{n\to+\infty}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$$

SOL. — Let us begin with proving that, for every $x \in [-\pi/4, \pi/4]\backslash\{0\}$,

$$\cos x \leq \frac{x}{\sin x} \leq 1 \tag{4.4.1}$$

Indeed, the area of the triangle $APO$ is $\frac{\sin x}{2}$, the area of the circle sector $APO$ is the area of the circle multiplied




by $\frac{x}{2\pi}$, that is, $\pi \cdot \dfrac{x}{2\pi} = \dfrac{x}{2}$, and the area of the triangle $AQO$ is $\frac{\tan x}{2}$. Therefore

$$\frac{\sin x}{2} \leq \frac{x}{2} \leq \frac{\tan x}{2} \qquad \forall x \in [-\pi/4, \pi/4]$$

For every $x \in [-\pi/4, \pi/4] \setminus \{0\}$ we may devide by $\dfrac{\sin x}{2}$ (which is $> 0$) we get

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

i.e.

$$\cos x \leq \frac{\sin x}{x} \leq 1, \qquad x \in [-\pi/4, \pi/4] \setminus \{0\}$$

which coincides with (4.4.1). So, in particular

$$\cos \frac{1}{n} \leq \frac{\sin \frac{1}{n}}{\frac{1}{n}} \leq 1 \qquad \forall n \in \mathbb{N}.$$

We claim that $\lim\limits_{n\to+\infty} \cos \dfrac{1}{n} = 1$. If so, by the two policemen theorem we get the thesis. It remains prove the claim. Actually, since $0 < \cos \frac{1}{n} \leq 1$, for every $n \in \mathbb{N}$, one has

$$0 \leq 1 - \cos \frac{1}{n} \leq 1 - \left(\cos \frac{1}{n}\right)^2 = \left(\sin \frac{1}{n}\right)^2 \leq \frac{1}{n^2}.$$

Hence, once again using the two policemen theorem , we get that $1 - \cos \dfrac{1}{n} \to 0$, i.e. $\lim\limits_{n\to+\infty} \cos \dfrac{1}{n} = 1$.

## 4.5. Principal infinities

As we have seen, it is important to compare (in the sense of Definition 4.4.2) different quantities to determine the biggest one. Among the fundamental quantities going to $+\infty$ there are exponential (with base $>1$), power (with exponent $>0$) and logarithm (with base $b>1$). It is clear that

$$a^n \gg_{+\infty} b^n, \iff a > b > 1; \quad n^\alpha \gg_{+\infty} n^\beta, \iff \alpha > \beta > 0.$$

Logarithms are not "ordered" because

$$\log_b n = (\log_b a)(\log_a n),$$

therefore $\log_b n \asymp_{+\infty} \log_a n$. The comparison between these quantities is not at all trivial:

**Proposition 4.5.1**

$$a^n \gg_{+\infty} n^\alpha \gg_{+\infty} \log_b n, \ \forall a > 1, \ \alpha > 0, \ b > 1.$$

Proof. We will limit to prove that $a^n \gg_{+\infty} n^\alpha$ for $\alpha > 0$. We first assume $\alpha < 1$. Notice that, writing $a = 1+h$ with $h > 0$, by binomial formula we have

$$(1+h)^n = \sum_{k=0}^{n} \binom{n}{k} h^k 1^{n-k} = \binom{n}{0} h^0 + \binom{n}{1} h^1 + \sum_{k=2}^{n} \binom{n}{k} h^k = 1 + nh + \sum_{k=2}^{n} \binom{n}{k} h^k \geqslant 1 + nh$$

being $h \geqslant 0$. This remarkable elementary inequality is called *Bernoulli inequality*. By this we have

$$\frac{a^n}{n^\alpha} \geqslant \frac{1+nh}{n^\alpha} = \frac{1}{n^\alpha} + hn^{1-\alpha} \longrightarrow +\infty$$

because $1 - \alpha > 0$.

We now pass to the case $\alpha \geqslant 1$. We reduce this to the case $\alpha < 1$ by writing

$$\frac{a^n}{n^\alpha} = \left(\frac{a^{\frac{n}{2\alpha}}}{n^{1/2}}\right)^{2\alpha} = \left(\frac{(a^{1/2\alpha})^n}{n^{1/2}}\right)^{2\alpha} =: \left(\frac{b^n}{n^{1/2}}\right)^{2\alpha},$$

where $b := a^{1/2\alpha} > 1$, being $a > 1$. Then $b^n \gg_{+\infty} n^{1/2}$ by previous case, so

$$\frac{b^n}{n^{1/2}} \longrightarrow +\infty, \implies \frac{a^n}{n^\alpha} = \left(\frac{b^n}{n^{1/2}}\right)^{2\alpha} \longrightarrow +\infty. \quad \square$$

Example 4.5.2. *Compute*

$$\lim_{n\to+\infty} \frac{2^n - n^2}{3^n - n^{100}2^n}.$$

Sol. — Consider the numerator first. Being $2^n \gg_{+\infty} n^2$ we have

$$N := 2^n - n^2 = 2^n \left(1 - \frac{n^2}{2^n}\right) = 2^n \cdot 1_n,$$

where, by shortness we set $1_n := 1 - \frac{n^2}{2^n} \longrightarrow 1$. In the denominator, apparently $n^{100}2^n$ is greater than $3^n$. This is false because

$$\frac{n^{100}2^n}{3^n} = \frac{n^{100}}{\left(\frac{3}{2}\right)^n} \overset{\left(\frac{3}{2}\right)^n \gg n^{100}}{\longrightarrow} 0, \implies 3^n - n^{100}2^n = 3^n\left(1 - \frac{n^{100}2^n}{3^n}\right) = 3^n \cdot 1_n.$$

Therefore

$$\frac{N}{D} = \frac{2^n \cdot 1_n}{3^n \cdot 1_n} = \left(\frac{2}{3}\right)^n \cdot 1_n \longrightarrow 0. \quad \square$$

Example 4.5.3. *Compute for $a > 0$ the*

$$\lim_{n\to+\infty} \frac{a^n - n^2a^{-n} + (-1)^n n^4}{a^{2n} + n^3}.$$

Sol. — It's better to treat separately the numerator and denominator. A the numerator seems evident that the first exponential dominates when $a > 1$ while, noticed that $n^2a^{-n} = n^2\left(\frac{1}{a}\right)^n$ it should be the second term to dominate when $a < 1$ (because $\frac{1}{a} > 1$). Thus, we should treat three different cases: $a > 1$, $a = 1$ and $a < 1$. If $a < 1$ we could write

$$N = -n^2a^{-n}\left(1 - \frac{a^n}{n^2a^{-n}} - \frac{(-1)^n n^4}{n^2a^{-n}}\right) = -n^2a^{-n}\left(1 - \frac{a^{2n}}{n^2} - (-1)^n\frac{n^2}{\left(\frac{1}{a}\right)^n}\right).$$

Now:

$$a^{2n} \longrightarrow 0,\ (a < 1), \implies \frac{a^{2n}}{n^2} \longrightarrow 0.$$

Moreover, being $a < 1$, that is $\frac{1}{a} > 1$, we have

$$\left(\frac{1}{a}\right)^n \gg n^2, \implies \frac{n^2}{\left(\frac{1}{a}\right)^n} \longrightarrow 0, \implies (-1)^n\frac{n^2}{\left(\frac{1}{a}\right)^n} \longrightarrow 0,\ \text{(bounded}\cdot\text{infinitesimal)}$$

hence $N = -n^2a^{-n} \cdot 1_n$. In the case $a = 1$ we have

$$N = 1 - n^2 + (-1)^n n^4 = (-1)^n n^4\left(1 + (-1)^n\frac{1}{n^4} - (-1)^n\frac{1}{n^2}\right) = (-1)^n n^4 \cdot 1_n,$$

again by the rule bounded·infinitesimal = infinitesimal. Finally, if $a > 1$ the dominating term is $a^n$ and indeed

$$N = a^n\left(1 - \frac{n^2a^{-n}}{a^n} + (-1)^n\frac{n^4}{a^n}\right) = a^n\left(1 - \frac{n^2}{a^{2n}} + (-1)^n\frac{n^4}{a^n}\right) = a^n \cdot 1_n$$

being $a^{2n} = (a^2)^n \gg n^2$ $(a > 1)$ and $a^n \gg n^4$ while $(-1)^n\frac{n^4}{a^n} \longrightarrow 0$ by the rule bounded·null=null. Summarizing,

$$N = \begin{cases} -n^2a^{-n} \cdot 1_n, & a < 1, \\ (-1)^n n^4 \cdot 1_n, & a = 1, \\ = a^n \cdot 1_n, & a > 1. \end{cases}$$

Also for the denominator we have the cases $a < 1$, $a = 1$, $a > 1$.

$$D = \begin{cases} n^3\left(1+\frac{a^{2n}}{n^3}\right) = n^3 \cdot 1_n,\ (a^{2n} \longrightarrow 0), & a < 1, \\ n^3\left(1+\frac{1}{n^3}\right) = n^3 \cdot 1_n, & a = 1, \\ a^{2n}\left(1+\frac{n^3}{a^{2n}}\right) = a^{2n} \cdot 1_n,\ (a^{2n} = (a^2)^n \gg n^3), & a > 1. \end{cases}$$

Finally,

$$\frac{N}{D} = \begin{cases} \frac{-n^2 a^{-n} \cdot 1_n}{n^3 \cdot 1_n} = -\frac{\left(\frac{1}{a}\right)^n}{n} \cdot 1_n \overset{\left(\frac{1}{a}\right)^n \gg n}{\longrightarrow} -\infty, & a < 1, \\ \frac{(-1)^n n^4 \cdot 1_n}{n^3 \cdot 1_n} = (-1)^n n \cdot 1_n,\ \text{doesn't exists}, & a = 1, \\ \frac{a^n \cdot 1_n}{a^{2n} \cdot 1_n} = \frac{1}{a^n} \cdot 1_n \longrightarrow 0, & a > 1. \end{cases} \qquad \square$$

Consider a limit of type

$$\lim_{n\to+\infty} a_n^{b_n}.$$

Here $(a_n) \subset ]0, +\infty[$ and $(b_n) \subset \mathbb{R}$. Notice that

$$a_n^{b_n} = a^{\log_a(a_n^{b_n})} = a^{b_n \log_a(a_n)},\ (a \neq 1,\ a > 0).$$

Choosing for instance $a > 1$, and accepting the *continuity of exponentials* (we will treat this topic later) we have

$$\text{if } \lim_{n\to+\infty} b_n \log_a(a_n) =: \ell, \implies \begin{cases} +\infty, & \text{when } \ell = +\infty, \\ a^\ell, & \text{when } \ell \in \mathbb{R}, \\ 0, & \text{when } \ell = -\infty. \end{cases}$$

The quantity $b_n \log_a(a_n)$ may lead to an indeterminate form of type $\infty \cdot 0$ or $0 \cdot \infty$. We have

$$b_n \log_a(a_n) = \infty \cdot 0, \iff \begin{cases} b_n \longrightarrow \pm\infty, \\ \log_a a_n \longrightarrow 0, \end{cases} \iff \begin{cases} b_n \longrightarrow \pm\infty, \\ a_n = a^{\log_a a_n} \longrightarrow 1, \end{cases}$$

or

$$b_n \log_a(a_n) = 0 \cdot \infty, \iff \begin{cases} b_n \longrightarrow 0, \\ \log_a a_n \longrightarrow \pm\infty, \end{cases} \iff \begin{cases} b_n \longrightarrow 0, \\ a_n \longrightarrow 0+,\ \vee\ a_n \longrightarrow +\infty. \end{cases}$$

Conclusion:

$$1^{\pm\infty}, \quad (0+)^0, \quad (0+)^{+\infty}$$

are **indeterminate forms**. The identity

$$a_n^{b_n} = a^{b_n \log_a a_n},$$

is the way to reduce this form to products.

EXAMPLE 4.5.4. *Compute*

$$\lim_{n\to+\infty}\left(\frac{1}{n}\right)^{\frac{\log_2 n}{n}}.$$

SOL. — We know that $n \gg_{+\infty} \log_2 n$, so $\frac{\log_2 n}{n} \longrightarrow 0$ while $\frac{1}{n} \longrightarrow 0+$: we have the indeterminate form $(0+)^0$. Now,

$$\left(\frac{1}{n}\right)^{\frac{\log_2 n}{n}} = 2^{\frac{\log_2 n}{n}\log_2 n} = 2^{\frac{(\log_2 n)^2}{n}} = 2^{\left(\frac{\log_2 n}{n^{1/2}}\right)^2} \longrightarrow 2^0 = 1,$$

because $n^{1/2} \gg_{+\infty} \log_2 n$. □

## 4.6. Monotonic sequences

In general, a sequence $(a_n) \subset \mathbb{R}$ might not have any limit (example: $(-1)^n$). However, for certain sequences it seems natural they converge:

**Definition 4.6.1**

We say that a sequence $(a_n)$ is **increasing** (notation: $a_n \nearrow$) if

$$a_{n+1} \geqslant a_n, \ \forall n \in \mathbb{N}.$$

Similarly, **decreasing** sequences are defined. An increasing/decreasing sequence is called **monotonic**.

An essential result is the

**Theorem 4.6.2**

Every monotonic sequence has always a limit. In particular:

$$\text{If } a_n \nearrow, \implies \exists \lim_n a_n = \sup\{a_n \ : \ n \in \mathbb{N}\} \in \mathbb{R}\cup\{+\infty\}.$$

$$\text{If } a_n \searrow, \implies \exists \lim_n a_n = \inf\{a_n \ : \ n \in \mathbb{N}\} \in \mathbb{R}\cup\{-\infty\}.$$

PROOF. Consider, for example, the case $a_n \nearrow$ (the other is left as exercise) and let

$$\ell := \sup\{a_n \ : \ n \in \mathbb{N}\}.$$

There are two possibilities: i) $\ell \in \mathbb{R}$, ii) $\ell = +\infty$.

i) $\ell \in \mathbb{R}$. Let $\varepsilon > 0$. By characteristic properties of sup, there exists $N = N \in \mathbb{R}$ such that

$$\ell - \varepsilon \leqslant a_N \leqslant \ell.$$

Being $a_n \nearrow$ we have

$$\ell - \varepsilon \leqslant a_N \leqslant a_n \leqslant \ell, \ \forall n \geqslant N, \implies |a_n - \ell| \leqslant \varepsilon, \ \forall n \geqslant N,$$

and this is nothing but the definition $a_n \longrightarrow \ell$.

ii) $\ell = +\infty$. In this case, $\{a_n \;:\; n \in \mathbb{N}\}$ is upper unbounded. Therefore, for any $K > 0$, we find an $N$ such that
$$a_N \geqslant K.$$
Being $a_n \nearrow$ we have
$$a_n \geqslant a_N \geqslant K, \; \forall n \geqslant N,$$
and this is nothing but the definition $a_n \longrightarrow +\infty$. □

A remarkable application of this result is to limit

**Proposition 4.6.3**

$$\exists \lim_{n\to+\infty} \left(1 + \frac{1}{n}\right)^n =: e \in ]2,3[. \tag{4.6.1}$$

Proof. We prove that $\left(1+\frac{1}{n}\right)^n$ is increasing. Indeed, according to Newton's formula, [1]
$$\left(1 + \frac{1}{n+1}\right)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1^k}{(n+1)^k} = 2 + \sum_{k=2}^{n+1} \frac{(n+1)!}{(n+1)^k(n+1-k)!}\frac{1}{k!}.$$
Notice that, for every $k = 2, \dots, n+1$,
$$\frac{(n+1)!}{(n+1)^k(n+1-k)!} = \frac{(n+1)n(n-1)\cdots(n+2-k)}{(n+1)(n+1)(n+1)\cdots(n+1)} = \frac{n(n-1)\cdots(n+2-k)}{(n+1)(n+1)\cdots(n+1)}$$
$$= \left(1 - \frac{1}{n+1}\right)\cdots\left(1 - \frac{k-1}{n+1}\right) > \left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) = \frac{n!}{n^k(n-k)!}.$$
Therefore
$$\left(1 + \frac{1}{n+1}\right)^{n+1} \geqslant 2 + \sum_{k=2}^{n+1} \frac{n!}{n^k(n-k)!}\frac{1}{k!} > 2 + \sum_{k=2}^{n} \frac{n!}{n^k(n-k)!}\frac{1}{k!} = \left(1 + \frac{1}{n}\right)^n,$$
thus $\left(1+\frac{1}{n}\right)^n$ is increasing, which in turn (by Theorem 4.6) implies that there exists the limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sup\left\{\left(1+\frac{1}{n}\right)^n, n \in \mathbb{N}\backslash\{0\}\right\} =: e$.[2] Since $\left(1+\frac{1}{n}\right)^n > 2$ for every $n > 1$, we have $e > 2$. Moreover,
$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k}{n}\right)\frac{1}{k!} < \sum_{k=0}^{n}\frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \sum_{k=4}^{n}\frac{1}{k!}.$$

---

[1]Newton's formula, for all $a, b \in \mathbb{R}$ and and all natural numbers $p \geq 1$, reads
$$(a+b)^p = \sum_{h=0}^{p}\binom{p}{h} a^{p-h}b^h$$

[2]In principle, the sequence might be unbounded, in which case $e$ should be interpreted as equal to $+\infty$. However, in what follows we show that this is not the case, for the sequence is bounded by 3.

Now, for $k \geqslant 2$, one has $k! = 1 \cdot 2 \cdot 3 \cdots k > 1 \cdot 2 \cdot 2 \cdots 2 = 2^{k-1}$, so that

$$\sum_{k=4}^{n} \frac{1}{k!} \leqslant \sum_{k=4}^{n} \frac{1}{2^{k-1}} = \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{n-1}} < \frac{1}{4},$$ [3]

thus

$$\left(1 + \frac{1}{n}\right)^n < 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{4} = \frac{35}{12}$$

Since $e$ is the supremum of $\left\{\left(1+\frac{1}{n}\right)^n, \quad n \in \mathbb{N}\right\}$ one has $e \leq \frac{35}{12} < 3$. □

EXAMPLE 4.6.4. *Compute*

$$\lim_{n\to+\infty} \left(1 - \frac{1}{n}\right)^n.$$

SOL. — We have

$$\left(1 - \frac{1}{n}\right)^n = \left(\frac{n-1}{n}\right)^n = \frac{1}{\left(\frac{n}{n-1}\right)^n} = \frac{1}{\left(1 + \frac{1}{n-1}\right)^n} = \frac{1}{\left(1 + \frac{1}{n-1}\right)^{n-1}\left(1 + \frac{1}{n-1}\right)} \longrightarrow \frac{1}{e \cdot 1} = \frac{1}{e}.$$ □

### 4.7. The concept of Mathematical Model

*Sequences* arise in a natural way in Mathematical Modelling, that is, when we meet stylized representations of some real system. Even if Mathematical Modelling is not a strict goal of this course, it is nevertheless useful to present some motivating example that gives some insight to future developments. Here we will limit to a few examples, chosen for their simplicity and not for their generality.

**4.7.1. Malthus model.** The Malthus model is an extremely simplified but yet interesting mathematical model to study the evolution of a *population*. This may be a human population, any biological population (such as cells, viruses, bacteria, etc), etc. Let us denote by $p_n$ the population at the $n$−th generation (here $n \in \mathbb{N}$). Depending on the population, $n$ may represent many different things (seconds, minutes, day, months, years, centuries, etc). To assess the population growth, we introduce the concept of *growth rate*

$$r_n := \frac{p_{n+1} - p_n}{p_n}. \tag{4.7.1}$$

This quantity represents the variation over the total of the population from an observation time $n$ to the next $n+1$. Equivalently,

$$p_{n+1} = (1 + r_n)p_n.$$

Since here $p_n$ represents the number of individuals at some time $n$, in particular $p_n \geqslant 0$, then necessarily $r_n \geqslant -1$. Actually, $r_n = -1$ would mean *complete extinction when we pass from time n to* $n+1$.

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[3]Indeed

$$\frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{n-1}} = \frac{1}{8}\left(1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^{n-4}\right) = \frac{1}{8}\frac{1 - \frac{1}{2^{n-3}}}{1 - \frac{1}{2}} < \frac{1}{4},$$

where we have used the identity $1 + x + \ldots x^q = \frac{1-x^{q+1}}{1-x}$, valid $x \in \mathbb{R}\backslash\{1\}$, $\forall q \in \mathbb{Z}$.

If the values of $p_n$ are known, $r_n$ can be determined. This is what happens when we use past observations of $p_n$. However, if we wish to predict future values of $p_n$ we need to do some *assumption* on how $r_n$ may depend on $n$ and on values of $p$. The simplest possible assumption is $r_n \equiv r$, that is

$$\frac{p_{n+1} - p_n}{p_n} = r, \; \forall n \in \mathbb{N}. \tag{4.7.2}$$

This is the *Malthus model*, introduced by the demographer Robert Malthus in 1798. In this case, the future values of $p_n$ given initial one, say $p_0$, can be easily determined. Indeed by (4.7.2) we have

$$p_n = (1+r)p_{n-1} = (1+r)^2 p_{n-2} = (1+r)^3 p_{n-3} = \ldots = (1+r)^n p_0.$$

The list $p_0, p_1, \ldots, p_n, p_{n+1}, \ldots$ is called *sequence*. In this case, it is quite easy to predict what happens to $p_n$ for a long future time $n$. Assume, for instance, that $-1 < r < 0$. To fix ideas, take $r = -\frac{1}{2}$. This means that $p_{n+1} - p_n$ is $-\frac{1}{2}p_n$ that is: first, population decreases ($p_{n+1} - p_n < 0$); second, the population halves. It seems evident that, in a long time, the population will be extinct. Precisely, for a fixed population threshold $\varepsilon > 0$ (small), we have

$$0 \leqslant p_n = \frac{1}{2^n} p_0 \leqslant \varepsilon, \iff 2^n \geqslant \frac{p_0}{\varepsilon}, \iff n \geqslant \log_2 \frac{p_0}{\varepsilon}, \iff n \geqslant N := \log_2 \frac{p_0}{\varepsilon}.$$

Thus, after time $N$ the population will remain below the threshold $\varepsilon$. This "game" can be repeated for every $\varepsilon > 0$ (of course the smaller is $\varepsilon$, the bigger is the initial time $N$ after which $a_n \leqslant \varepsilon$). This gives a clear idea that $p_n$ can be made small as we like provided we wait long enough. This is the idea behind the concept of

$$\lim_{n \to +\infty} p_n = 0.$$

If $r > 0$, as for example if $r = 1$, then $p_n = 2^n p_0$ thus $p_n$ becomes arbitrarily large when $n$ increases. Indeed, fixed $K > 0$ we see that

$$p_n \geqslant K, \iff 2^n \geqslant \frac{K}{p_0}, \iff n \geqslant \log_2 \frac{K}{p_0}, \iff n \geqslant N := \left[\log_2 \frac{K}{p_0}\right] + 1.$$

In this case, we would say

$$\lim_{n \to +\infty} p_n = +\infty.$$

**4.7.2. Logistic map.** Malthus model is simple but not realistic. If, for instance, we assume a positive growth (that is, $r > 0$) we have $p_n \longrightarrow +\infty$. If the environment has some physical limitations (such as space or food), this is manifestly impossible. A more reasonable model should have a *variable growth rate*, that is, a rate that depends on the population $p$, namely

$$r_n = r(p_n),$$

where $r = r(p)$ is a suitable modelling rate function. For example, if we estimate that a certain environment can be tolerating a population up to a certain maximal size $M$, we should expect something like

$$r(p) \begin{cases} > 0, & \text{if } p < M, \\ = 0, & \text{if } p = M, \\ < 0, & \text{if } p > M. \end{cases}$$

In this way, the population should decrease if it passes the maximal size, while it should be free to grow if the environment allows it, that is, below the maximal size. Another reasonable property is that *growth rate should decrease if the population increases*. The simplest possible function with these properties is a *linear function*

$$r(p) = r_0(M - p).$$

Plugging this into (4.7.1) we obtain the following equation

$$\frac{p_{n+1} - p_n}{p_n} = r_0(M - p_n),$$

that is

$$p_{n+1} = p_n \left(1 + r_0(M - p_n)\right).$$

By rearranging the terms and rescaling, this equation may be reduced to a one-parameter equation

$$\boxed{x_{n+1} = \rho x_n(1 - x_n),} \tag{4.7.3}$$

and under this form is known as **logistic map**. We might use directly this as a model equation, in the sense that here $x_n$ may represent the *percentage of site occupation*. We may notice that if $x_n \approx 0$ (in other words, if $x_n$ s very small respect to 1) then

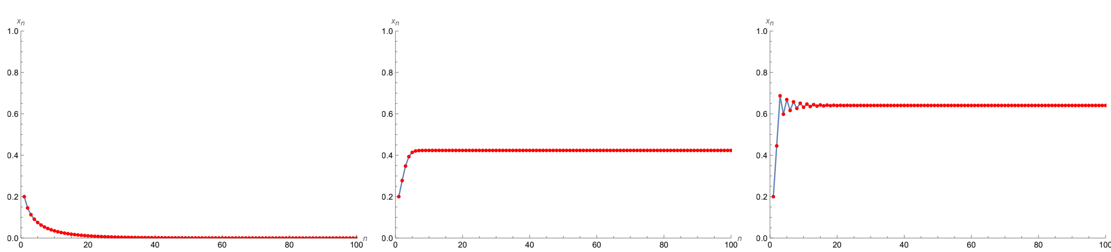
$$x_{n+1} \approx \rho x_n.$$

Thus, when $x_n$ is small (respect to 1) the population follows a Malthus model evolution, so $\rho$ plays the role of $1 + r_0$. In particular, $\rho = 1 + r_0 < 1$ means $r_0 < 0$ thus we may expect that the population will go to the extinction while for $\rho > 1$ ($r_0 > 0$) the population will grow. In this case, sooner or later the population will be no more negligible, then it will not follow the Malthus dynamics. It is at this point that the story becomes complicated.

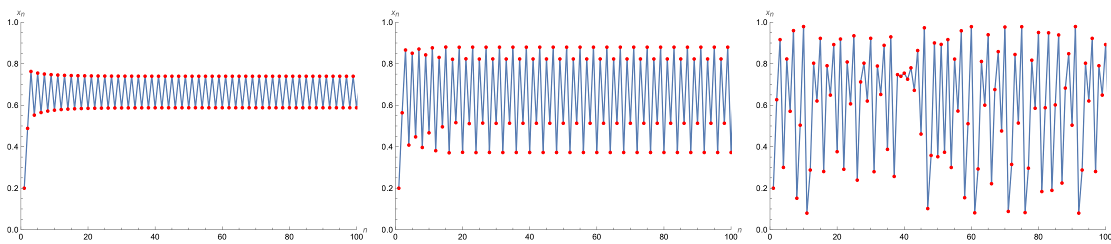
The behaviour of solutions to this equation may be extremely complex. The first problem is that there is no way to get a direct dependence of $x_n$ from the model parameters. For example:

$$\begin{aligned}
x_1 &= \rho x_0(1 - x_0), \\
x_2 &= \rho x_1(1 - x_1) = \rho^2 x_0(1 - x_0)(1 - \rho x_0(1 - x_0)), \\
x_3 &= \ldots
\end{aligned}$$

However, *recurrence equation* (4.7.3) may be used to determine the numeric plots of the sequence $x_n$. This can be easily implemented into a code that you may easily create with any simple Programming Language. In the following figures we show some of the plots for values of $x_n$ in function of $n$. Initial $x_0$ is conventionally fixed to $x_0 = 0.2$ (that is, 20% of the available space is occupied).

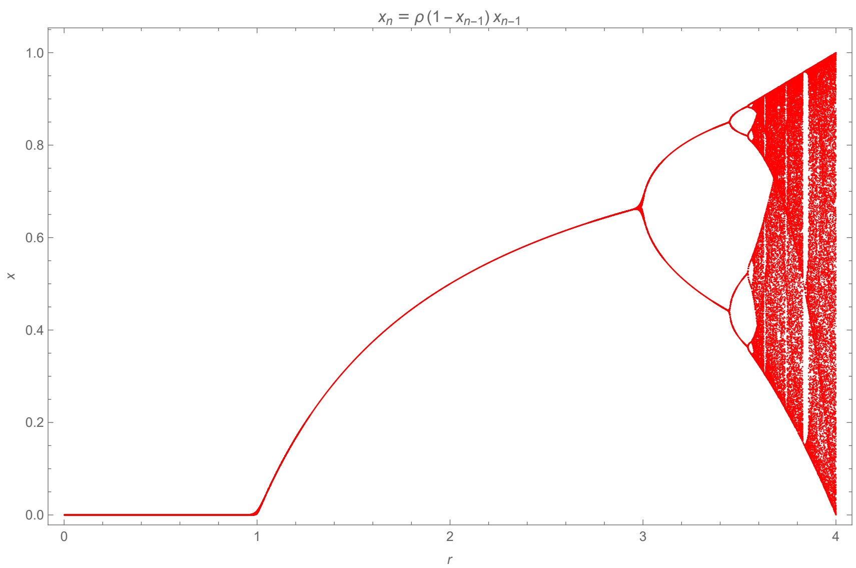


The first three figures show cases $\rho = 0.7, 1.7$ and $\rho = 2.7$. We see that, in the long time, $x_n \approx 0$ in the first case while in second and third case $x_n \approx \ell$ with $\ell \neq 0$. We would say that in these cases the population tends to an *equilibrium*, that is, a state that does not change in time.



This second set shows cases $\rho = 3.1, 3.5$ and $3.9$. In the first case, we see that, for $n$ big, $x_n$ oscillates around two values, that is $x_n \approx \ell_1$ and $x_{n+1} \approx \ell_2$ with $\ell_1 \neq \ell_2$. Population does not attain an equilibrium, yet the long time behaviour is regular. In the second case the situation gets more complicated: the long time behaviour of $x_n$ sees this oscillate around four distinct values. The last case is a completely different story: there is no regularity on the value $x_n$. This is an example of what is called *chaotic dynamics*.

To have a plastic view of the long time behaviour, the next figure is interesting. Here we plot, in function of $\rho$, the values around which $x_n$ oscillates for long times. For example: if $0 \leqslant \rho \leqslant 1$, since $x_n \approx 0$ for $n$ big, we plot 0.




We see that $x_n \approx \ell(\rho)$ for $0 \leqslant \rho \leqslant 3$. For $\rho$ slightly greater than 3 we see that $x_n$ oscillates around two values $\ell_1(\rho) < \ell_2(\rho)$, thus previous line $\ell(rho)$ "bifurcates" around $\rho = 3$ into two branches. Increasing $\rho$ we see that at some $\rho$ around $3.4\ldots$ the system oscillates around four values. Again, each of branches $\ell_1(\rho)$ and $\ell_2(\rho)$ bifurcates. This mechanism seems to become faster and faster and around $\rho = 3.6\ldots$ the system starts to have an irregular behaviour.

Of course, these are empirical observations based on a finite number of iterations of the logistic map to compute a relatively small number of $x_n$ (we computed $x_n$ for $n = 0, 1, \ldots, 100$). Nobody could say if these observations remain valid for larger $n$ (who knows about $x_{10^{10}}$?). Nonetheless, they suggest a number of hypotheses that could be verified. Despite its simplicity, the logistic map contains a lot of complexity. It is perhaps the simplest example of *chaotic dynamics*.

**4.7.3. Fibonacci numbers.** This model describes the evolution of an ideal population of rabbits. Precisely, the model describes $a_n$, number of couples of rabbits at $n-$th generation. The model assumptions are:

- Initially, there is a unique newborn couple of rabbits, that is, $a_0 = 1$.
- Each new born couple, after a generation becomes fertile, and after two a new couple is born.

Thus, $a_0 = 1$. At $n = 1$ there is the initial couple that starts mating. At generation $n = 2$ there is the initial couple, still mating, and a newborn couple: $a_2 = 2$. Next generation we have the initial couple, that one born at $n = 2$ and not starting to mate, and a newborn couple created by the first couple. Thus $a_3 = 3$. We need a general rule: at $n$ th generation, there are all couples of previous generation, that is $a_{n-1}$, plus a new couple for each of the couples of the $n - 2$ th generation, that is $a_{n-2}$. In conclusion

$$a_n = a_{n-1} + a_{n-2}. \tag{4.7.4}$$

So, we can compute some terms of this sequences, taking in mind that $a_0 = 1, a_1 = 1$:

$$1, 1, 2, , 3, 5, 8, 13, 21, 44, 65, 109, 174, 283, 457, .....$$

It is clear that the population grows very fast. But *how much fast*? Let us examine the ratio $q_n := \frac{a_n}{a_{n-1}}$ between an element $a_n$ and the previous one $a_{n-1}$, and let us bet that this ratio tend to a constant $q$ as $n$ tends to infinity. (Clearly, provided this constant is $> 1$ this means that we are conjecturing that the growth is exponential for large values of $n$). By the Fibonacci's relation (4.7.4) we have

$$q = \lim_{n\to\infty} q_n = \lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \lim_{n\to\infty} \frac{a_{n-1} + a_{n-2}}{a_{n-1}} = 1 + \lim_{n\to\infty} \frac{a_{n-2}}{a_{n-1}} = 1 + \lim_{n\to\infty} \frac{1}{q_{n-1}} = 1 + \frac{1}{q}$$

Hence, we get $q = 1 + \frac{1}{q}$, i.e. $q^2 - q - 1 = 0$, that is

$$q = \frac{1 \pm \sqrt{5}}{2}.$$

Hence, there are to candidates to mimic the Fibonacci sequence, namely

$$\widehat{a_n} = \left(\frac{1 + \sqrt{5}}{2}\right)^n, \quad \widetilde{a_n} = \left(\frac{1 - \sqrt{5}}{2}\right)^n.$$

Now, though the above sequences verify the conditions $\widehat{a_1} = 1, \widetilde{a_1} = 1$, neither $\widehat{a_0} = 1$ nor $\widetilde{a_0} = 1$ hold true. However, since the Fibonacci's relation (4.7.4) is linear, every linear combination of the above sequences, namely

$$b_n := c_1\widehat{a_n} + c_2\widetilde{a_n} = c_1 \left(\frac{1 + \sqrt{5}}{2}\right)^n + c_2 \left(\frac{1 - \sqrt{5}}{2}\right)^n,$$

for some constants $c_1, c_2$, verifies it. Let us determine the constants $c_1, c_2$ by imposing the initial relations $b_0 = 1, b_1 = 1$. This give

$$\begin{cases} c_1 + c_2 = 1, \\ c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2} = 1 \end{cases} \iff \begin{cases} c_1 + c_2 = 1, \\ \sqrt{5}(c_1 - c_2) = 1 \end{cases} \iff c_1 = \frac{1}{2} + \frac{1}{2\sqrt{5}}, \ c_2 = \frac{1}{2} - \frac{1}{2\sqrt{5}},$$

so that

$$b_n = \left(\frac{1}{2} + \frac{1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1}{2} - \frac{1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

As $n \longrightarrow \infty$ it is clear that being $\frac{1+\sqrt{5}}{2} > 1$ we have $\left(\frac{1+\sqrt{5}}{2}\right)^n \longrightarrow +\infty$, while, being $-1 \leqslant \frac{1-\sqrt{5}}{2} \leqslant 0$ we should have $\left(\frac{1-\sqrt{5}}{2}\right)^n \longrightarrow 0$. In other words, $b_n$ is asymptotic to $\left(1 + \frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n$, which we write as

$$b_n \sim \left(1 + \frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

Some more advanced theory –which is beyond the goals of the present course– tell us that the Fibonacci sequence $(a_n)$ is actually asymptotic to $(b_n)$, so that we can conclude that

$$a_n \sim \left(1 + \frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

as well.

**4.7.4. Interest rates.** Suppose we have a sum $s$ and we deposit it into a bank. After some time, the bank corresponds an *interest rate* $r$. Normally, a bank corresponds a rate $r$ on an annual basis. There are several different ways to compute the final premium. For instance, one could consider $s + rs = (1 + r)s$. Another possibility could be the following: divide the year in $n$ parts (for instance $n = 365$ dividing the year in days). Each interval of this subdivision, the bank corresponds a fraction $\frac{r}{n}$ of $r$. This means that after the first period we will have the sum

$$s + \frac{r}{n}s = \left(1 + \frac{r}{n}\right)s.$$

After a second period we will have

$$\left(1 + \frac{r}{n}\right)s + \frac{r}{n}\left(1 + \frac{r}{n}\right)s = \left(1 + \frac{r}{n}\right)\left(1 + \frac{r}{n}\right)s = \left(1 + \frac{r}{n}\right)^2 s.$$

Easily we can see see that after $n$ periods, therefore at the end of the entire period, we will have the sum

$$s_n := \left(1 + \frac{r}{n}\right)^n s.$$

The first natural question is: what number $n$ of periods maximize the premium? Clearly everything depends on $\left(1 + \frac{r}{n}\right)^n$. We notice immediately that

$$a_2 = \left(1 + \frac{r}{2}\right)^2 = 1 + 2\frac{r}{2} + \frac{r^2}{4} = 1 + r + \frac{r^2}{4} \geqslant 1 + r = a_1.$$

Similarly

$$a_3 = \left(1+\frac{r}{3}\right)^3 = 1+3\frac{r}{3}+3\frac{r^2}{9}+\frac{r^3}{27} = 1+r+\frac{r^2}{3}+\frac{r^3}{27} \geqslant 1+r+\frac{r^2}{4} = a_2.$$

This leads immediately to the following *conjecture*: *let* $a_n := \left(1+\frac{r}{n}\right)^n$*; then* $a_{n+1} \geqslant a_n$ *for every* $n$. This is true if $r > 0$, but it is not easy to prove it in general. Now, assuming $\left(1+\frac{r}{n}\right)^n \nearrow$ as $n \nearrow$, how much can be big this quantity? Does it increase to $+\infty$ or to some limit $\ell$ finite? It is clear that in the first case it would mean an infinite profit, and this sounds wrong ... We will see that there exists a special number, $e \in ]2,3[$, such that

$$\left(1+\frac{r}{n}\right)^n \longrightarrow e^r, \text{ as } n \longrightarrow +\infty,$$

The number $e$ is one of more important constant of Mathematical Analysis: the *Napier number*.

## 4.8. Exercises

EXERCISE 4.8.1. *Using the definition, show that*

1. $\lim_{n\to+\infty}\left(1+\frac{1}{2^n}\right) = 1.$
2. $\lim_{n\to+\infty}\frac{n}{n+1} = 1.$
3. $\lim_{n\to+\infty}\frac{n+2}{n+1} = 1.$
4. $\lim_{n\to+\infty}\frac{n+2}{n^2+3n} = 0.$
5. $\lim_{n\to+\infty}\frac{n+\sqrt{n}}{n-\sqrt{n}} = 1.$
6. $\lim_{n\to+\infty}\frac{1}{n^\alpha} = 0, \ (\alpha > 0).$
7. $\lim_{n\to+\infty}\left(\log_{10}(n+1)-\log_{10} n\right) = 0.$
8. $\lim_{n\to+\infty}\frac{2n^2+1}{n^2+n+1} = 2.$
9. $\lim_{n\to+\infty}\frac{2^n-2^{-n}}{2^n+2^{-n}} = 1.$
10. $\lim_{n\to+\infty}\left(n-\sqrt{n-1}\right) = +\infty.$
11. $\lim_{n\to+\infty}\frac{n^2+1}{n+1} = +\infty.$
12. $\lim_{n\to+\infty}\frac{1-n}{1+\sqrt{n}} = -\infty.$
13. $\lim_{n\to+\infty}\frac{n}{1-\sqrt{n}} = -\infty.$
14. $\lim_{n\to+\infty}\left(n+\frac{1}{n}\right) = +\infty$
15. $\lim_{n\to+\infty}\frac{3n-1}{2-2n} = -\frac{3}{2}.$

EXERCISE 4.8.2. *By using suitable subsequences, we show that the following sequences do not have a limit.*

1. $(-1)^n n^2$. 2. $\sin\frac{n\pi}{4}$. 3. $(-1)^n-(-1)^{n^2}$. 4. $\frac{1+(-1)^n}{2}$. 5. $(-2)^n$. 6. $(-1)^n\frac{3n+1}{n}$.

EXERCISE 4.8.3. *For each of the following sequences, find two policemen and compute the limit:*

1. $\frac{n+\cos n}{n+1}$. 2. $\frac{n^2+(-1)^n n}{n^2+1}$. 3. $\frac{(-1)^n n}{n^2+1}$. 4. $\frac{n\sin n}{n^2+1}$. 5. $\frac{\sqrt{n}\sin(n!)}{n+1}$.

EXERCISE 4.8.4. Compute

1. $\lim\limits_{n\to+\infty}\left(\dfrac{2-3n}{2n+1}\right)^3$. 2. $\lim\limits_{n\to+\infty}\left(\sqrt{n}-\sqrt[3]{n}\right)$. 3. $\lim\limits_{n\to+\infty}\dfrac{\sqrt[3]{n}+\sqrt[5]{n}}{\sqrt{n}-1}$.

4. $\lim\limits_{n\to+\infty}\sqrt[3]{n+a}-\sqrt[3]{n}$, $(a>0)$. 5. $\lim\limits_{n\to\infty}\left(\sqrt{2n^2+n-1}-n\sqrt{2}\right)$. 6. $\lim\limits_{n\to+\infty}\dfrac{((n+1)!)^2(2n)!}{(2(n+1))!(n!)^2}$.

7. $\lim\limits_{n\to+\infty}\dfrac{n!}{(n+1)!-n!}$. 8. $\lim\limits_{n\to+\infty}\dfrac{n+\sin n}{n-\cos n}$. 9. $\lim\limits_{n\to+\infty}(n+(-1)^n\sqrt{n})$.

10. $\lim\limits_{n\to+\infty}n^2\left(1-\sqrt[3]{\dfrac{n^2-1}{n^2+1}}\right)$. 11. $\lim\limits_{n\to+\infty}\dfrac{n^3+n^2\sin\frac{1}{n}}{n^3+1}$.

EXERCISE 4.8.5. *Discuss in function of the parameter $\alpha\in\mathbb{R}$ existence and value of*

1. $\lim\limits_{n\to+\infty}\dfrac{n^\alpha-n+1}{n^2+1}$. 2. $\lim\limits_{n\to+\infty}\sqrt{n^\alpha+5}-\sqrt{n+1}$. 3. $\lim\limits_{n\to+\infty}n^\alpha\left(\sqrt{n+1}-\sqrt{n}\right)$.

EXERCISE 4.8.6. *Compute*

1. $\lim\limits_{n\to+\infty}\dfrac{2^n}{(n+2)^n}$. 2. $\lim\limits_{n\to+\infty}\dfrac{2^n+n^5}{3^n-n^2}$. 3. $\lim\limits_{n\to+\infty}\dfrac{n^2 2^n+n^{10}-5^n}{n^2 5^n+10^n}$. 4. $\lim\limits_{n\to+\infty}\dfrac{n^7+2^n-5^n}{n^{5000}-3^n}$.

5. $\lim\limits_{n\to+\infty}\dfrac{2^{n^2}}{n^4+1}$. 6. $\lim\limits_{n\to+\infty}\dfrac{2n+(\sin n)(\log_2 n)}{n}$. 7. $\lim\limits_{n\to+\infty}\dfrac{(\log_2 n)^4+(\log_4 n)^2}{\sqrt[10]{n}+1}$.

EXERCISE 4.8.7 (★). *Discuss, in dependence of the parameter $a>0$, existence and value of*

1. $\lim\limits_{n\to+\infty}\dfrac{n^2a^n-n2^n+n^3}{n^8 3^{-n}+n^2 2^n+n^3(\sin n)^n}$. 2. $\lim\limits_{n\to+\infty}\dfrac{2^n n^a-2^n n^3+2^{-n}\cos n^n}{2^n n^4-2^{-n}n^8+n^{10}}$.

3. $\lim\limits_{n\to+\infty}\dfrac{a^n+n^2 3^n}{(-3)^n+n^2 2^n-a^{2n}}$. 4. $\lim\limits_{n\to+\infty}\dfrac{a^n-n^4 3^{2n}+3^n\cos(n!)}{n^4 9^n+9^{-n}4^{2n}-n^9}$.

5. $\lim\limits_{n\to+\infty}\dfrac{a^n-n2^n+\frac{4^n}{n!}\sin n}{a^n+n2^n-3^n}$. 6. $\lim\limits_{n\to+\infty}\dfrac{2^n n^\alpha-2^n n^3+(2\cos n)^{-n}}{2^n n-2^{-n}n^3+n^3}$.

7. $\lim\limits_{n\to+\infty}\dfrac{n^4 4^n-(4a)^n+4^n n^2\cos(n!)}{n^{-n}-n^4 2^{3n}+12^n}$. 8. $\lim\limits_{n\to+\infty}\dfrac{n^{-n}-3^{2n}n^{12}+12^n}{6^n n^2\sin(n!)-(3a)^n+n^3 6^n}$.

EXERCISE 4.8.8. *Compute, in function of the parameter $a\in\mathbb{R}$,*

1. $\lim\limits_{n\to+\infty}\left[\left(\dfrac{2a^2}{a^2+1}\right)^n-n^{-5a}\right]$. 2. $\lim\limits_{n\to+\infty}\dfrac{a^n-\log n}{a^n+2^n}$.

EXERCISE 4.8.9 (★). *Order the following quantities with respect the symbol $\gg$:*

$$n\sqrt{n},\quad n^{2^n},\quad 2^{\log_2 n+\log_4 n},\quad 2^{2^n},\quad n^{1+\frac{1}{\sqrt{\log_2 n}}}.$$

Exercise 4.8.10. *Reducing to the limit of e, compute*

$$1.\quad \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{3n}.\quad 2.\quad \lim_{n\to+\infty}\left(1+\frac{1}{n^2}\right)^{n^3}.\quad 3.\quad \lim_{n\to+\infty}\left(1+\frac{1}{n+k}\right)^{n},\ (k\in\mathbb{N}).$$

$$4.\quad \lim_{n\to+\infty}\frac{(n+1)^n}{n^{n+1}}.\qquad 5.\quad \lim_{n\to+\infty}\frac{1}{n}\left(\frac{n+3}{n+2}\right)^{n}.\quad 6.\quad \lim_{n\to+\infty}\frac{n^{2n}}{(n+1)^{2n}}.$$

Exercise 4.8.11. *Compute*

$$1.\ (\star)\quad \lim_{n\to+\infty}\frac{(n-1)^{n+\frac{1}{\log n}}}{(n+1)^{\sqrt{1+n^2}}}.\quad 2.\ (\star\star)\quad \lim_{n\to+\infty}\frac{n!}{n^{n/2}}$$

Exercise 4.8.12. *Prove that*

$$n^n \gg n! \gg a^n,\ \forall a>1.$$

CHAPTER 5

# Limit

## 5.1. Definition of the limit of a function

Once we are acquainted with the limits of sequences, we can extend this concept to the general case of functions of a real variable, $f = f(x)$, $x \in D \subset \mathbb{R}$. In other words, we want to give a meaning

$$\lim_{x \to \xi} f(x) = \ell,$$

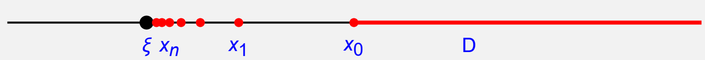
where $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$. To define the limit, $\xi$ cannot be any point. Indeed, since we wish to check what happens to $f(x)$ when $x$ is close to $\xi$, we need $f(x)$ be defined for $x$ close to $\xi$. In other words, $\xi$ should be approachable by points of $D$. This leads to the following important

**Definition 5.1.1: accumulation point**

Let $D \subset \mathbb{R}$ and $\xi \in \mathbb{R} \cup \{\pm\infty\}$. We say that $\xi$ is an **accumulation point for $D$** if

$$\exists\ (x_n) \subset D \backslash \{\xi\}\ :\ x_n \longrightarrow \xi. \tag{5.1.1}$$

$\mathrm{Acc}(D)$ denotes the set of accumulation points of $D$.




Some useful remarks on this definition:

- *accumulation points of $D$ are not necessarily elements of $D$*: for example

$$a \in \mathrm{Acc}(]a, b[)\ \ (\text{take } x_n = a + \tfrac{1}{n})$$

- In the definition, we required the existence of $(x_n) \subset D\backslash\{\xi\}$ such that $x_n \longrightarrow \xi$. In particular, $x_n \neq \xi$ for all $n$. In this way, we exclude **isolated points** of $D$ being accumulation points: for example, if $D = \{\xi\}$ then $\mathrm{Acc}(D) = \emptyset$.
- $\pm\infty$ may be accumulation points of $D$. In this case, the restriction $(x_n) \subset D\backslash\{\xi\} \equiv D$ because $D \subset \mathbb{R}$. To say that $+\infty \in \mathrm{Acc}(D)$ is equivalent to say that $\sup D = +\infty$ as well as $-\infty \in \mathrm{Acc}(D)$ iff $\inf S = -\infty$.

Example 5.1.2. We have

- $0 \in \mathrm{Acc}\left(\left\{\frac{1}{n}\ :\ n \in \mathbb{N},\ n \geqslant 1\right\}\right)$: indeed, $\left(\frac{1}{n}\right) \subset D := \left\{\frac{1}{n}\ :\ n \in \mathbb{N},\ n \geqslant 1\right\}$, $\frac{1}{n} \longrightarrow 0$ and $\frac{1}{n} \neq 0$ for any $n$. More difficult is to show that 0 is the unique accumulation point for $D$ (even if it should be intuitively clear!).
- $\mathrm{Acc}(\mathbb{R}) = \mathbb{R} \cup \{\pm\infty\}$.

- $\mathrm{Acc}(\mathbb{N}) = \{+\infty\}$, $\mathrm{Acc}(\mathbb{Z}) = \{\pm\infty\}$.
- $\mathrm{Acc}(\{0\} \cup [1,2]) = [1,2]$.

We are now ready for the

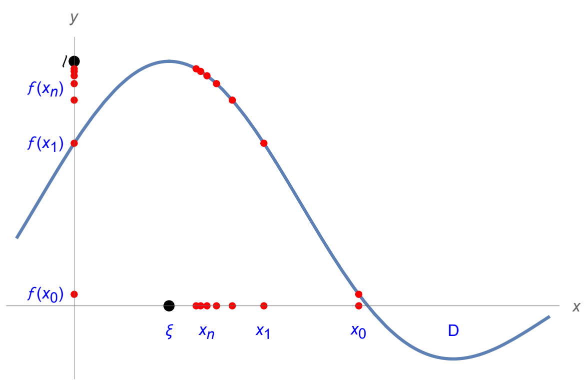
**Definition 5.1.3**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\xi \in \mathrm{Acc}(D)$. We say that

$$\exists \lim_{x\to\xi} f(x) = \ell \in \mathbb{R} \cup \{\pm\infty\}$$

if the following property holds true:

$$\forall\, (x_n) \subset D\backslash\{\xi\},\ x_n \longrightarrow \xi, \implies f(x_n) \longrightarrow \ell. \tag{5.1.2}$$



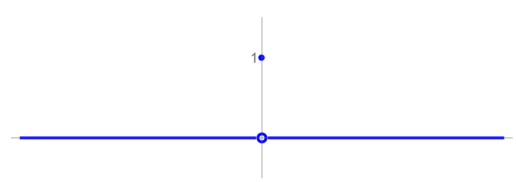

Remark 5.1.4 (Important!). In (5.1.2) we consider a sequence $(x_n) \subset D$ (this to have $f(x_n)$ defined) such that $x_n \neq \xi$, this regardless if $\xi \in D$ or not. This seems to be a detail but it is not! Indeed: first of all, *function $f$ might not be defined at $\xi$*. For example, this is always the case when $\xi = \pm\infty$, which is allowed by the definition. For example, it makes sense to consider

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

Here $f(x) := \frac{\sin x}{x}$ is defined on $D = \mathbb{R}\backslash\{0\}$, clearly $0 \in \mathrm{Acc}D$ but $f(0)$ is not defined.

Moreover, even if $f$ is defined at $\xi$, *the value at $f(\xi)$, may not have anything to share with the* $\lim_{x\to\xi} f(x)$. This is a subtle point, we show it by an example. Consider the function

$$f(x) = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0. \end{cases}$$




What is the reasonable guess on $\lim_{x\to 0} f(x)$? Does it exist? If yes, what should its value be? Intuitively,since for $x$ close to 0 (but different from 0), we have $f(x) = 0$, then it is reasonable to conclude that $\lim_{x\to 0} f(x) = 0$.

And indeed this is what happens: if $(x_n) \subset D\backslash\{0\}$ and $x_n \longrightarrow 0$, then $f(x_n) \equiv 0$ (being $x_n \neq 0$), so $f(x_n) \equiv 0 \longrightarrow 0$. According to Definition 5.1, we conclude $\lim\limits_{x\to 0} f(x) = 0$, as expected.

Imagine now that we modify (5.1.2) as follows:

$$\forall\, (x_n) \subset D,\ x_n \longrightarrow \xi, \implies f(x_n) \longrightarrow \ell. \tag{5.1.3}$$

In this case, we would conclude that $\lim_{x\to 0} f(x)$ would not exist! Indeed: take $(x_n) \subset D$. As before, if $x_n \neq 0$ we have $f(x_n) \equiv 0 \longrightarrow 0$. In this case, however, we can also take $x_n \equiv 0$ is contained in $D$ and $x_n \longrightarrow 0$, but now $f(x_n) = f(0) \equiv 1 \longrightarrow 1$. Thus (5.1.3) would not be verified. □

The notion of limit can be given independently of sequences, for the following characterization holds true:

**Proposition 5.1.5**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\xi \in \mathrm{Acc}(D) \cap \mathbb{R}$. Then the following equivalences hold true:

$$\lim_{x\to\xi} f(x) = \ell \in \mathbb{R} \iff \forall \epsilon > 0\ \exists \delta > 0 :\ \ l - \epsilon < f(x) < l + \epsilon \quad \forall x \in ]\xi - \delta, \xi + \delta[ \cap D\backslash\{\xi\}$$

$$\lim_{x\to\xi} f(x) = +\infty \iff \forall K \in \mathbb{R}\ \ \exists \delta > 0 :\ \ f(x) > K \quad \forall x \in ]\xi - \delta, \xi + \delta[ \cap D\backslash\{\xi\}$$

$$\lim_{x\to\xi} f(x) = -\infty \iff \forall K \in \mathbb{R}\ \ \exists \delta > 0 :\ \ f(x) < K \quad \forall x \in ]\xi - \delta, \xi + \delta[ \cap D\backslash\{\xi\}$$

Let $+\infty \in \mathrm{Acc}(D)$ (i.e., let $D$ be not upper bounded). Then the following equivalences hold true:

$$\lim_{x\to+\infty} f(x) = \ell \in \mathbb{R} \iff \forall \epsilon > 0\ \exists M :\ \ l - \epsilon < f(x) < l + \epsilon \quad \forall x \in D \cap ]M, +\infty[$$

$$\lim_{x\to+\infty} f(x) = +\infty \iff \forall K \in \mathbb{R}\quad \exists M :\ \ f(x) > K \quad \forall x \in D \cap ]M, +\infty[$$

$$\lim_{x\to+\infty} f(x) = -\infty \iff \forall K \in \mathbb{R}\ \ \exists M :\quad f(x) < K \quad \forall x \in D \cap ]M, +\infty[$$

Let $+\infty \in \mathrm{Acc}(D)$ (i.e., let $D$ be not lower bounded). Then the following equivalences hold true:

$$\lim_{x\to-\infty} f(x) = \ell \in \mathbb{R} \iff \forall \epsilon > 0\ \exists M :\ \ l - \epsilon < f(x) < l + \epsilon \quad \forall x \in D \cap ] - \infty, M[$$

$$\lim_{x\to-\infty} f(x) = -\infty \iff \forall K \in \mathbb{R}\quad \exists M :\ \ f(x) > K \quad \forall x \in D \cap ] - \infty, M[$$

$$\lim_{x\to-\infty} f(x) = -\infty \iff \forall K \in \mathbb{R}\ \ \exists M :\quad f(x) < K \quad \forall x \in D \cap ] - \infty, M[$$

PROOF. Let us prove only the first equivalence, the other ones being left as an exercise. Namely, we have to prove that

$$\lim_{x\to\xi} f(x) = \ell \in \mathbb{R} \Longleftrightarrow \forall\epsilon > 0 \ \exists\delta > 0 : \ \ l - \epsilon < f(x) < l + \epsilon \quad \forall x \in ]\xi - \delta, \xi + \delta[$$

Let us begin with proving the implication " $\Longrightarrow$ ". We proceed by contradiction. That is, we deny the thesis, so obtaining

$$\exists\bar{\epsilon} : \forall\delta > 0 \ \ \exists x_\delta \in ]\xi - \delta, x + \delta[\cap D \ \text{ and } \ f(x_\delta) \notin ]l - \bar{\epsilon}, l + \bar{\epsilon}[.$$

Since $\delta$ is arbitrary we can choose, for every $n \in \mathbb{N}\backslash\{0\}$, $\delta = \frac{1}{n}$, so that

$$\exists x_n \in \left]\xi - \frac{1}{n}, \xi + \frac{1}{n}\right[ \cap D \ \text{and} f(x_n) \notin ]l - \bar{\epsilon}, l + \bar{\epsilon}[.$$

Notice that $x_n \to \xi$. Hence the so-constructed sequence $(x_n)$ is such that $f(x_n) \notin ]l - \bar{\epsilon}, l + \bar{\epsilon}[$ for every $n > 0$, which contradicts the thesis $\lim_{x\to\xi} f(x) = \ell$. The implication $\Longrightarrow$ has been proved.

Let us now prove the implication " $\Longleftarrow$ ". Let $(x_n)$ be a sequence in $D\backslash\xi$ such that $x_n \to \xi$. By hypothesis, for every $\epsilon > 0$, there is a $\delta > 0$ such that $f(x) \in ]l - \epsilon, l + \epsilon[$ for all $x \in ]\xi - \delta, \xi + \delta[\cap D\backslash\{\xi\}$. Since $x_n \to \xi$, there exists $N > 0$ so that $\forall n > N$ one has $x_n \in ]\xi - \delta, \xi + \delta[\cap D\backslash\{\xi\}$. Hence, by hypothesis, $f(x_n) \in ]l - \epsilon, l + \epsilon[$. Therefore the sequence $(x_n)$ verifies $f(x_n) \to \ell$. By the arbitrariness of the sequence $(x_n)$, we obtain $\lim_{x\to\xi} f(x) = \ell$, so the proof is concluded. □

When $x \longrightarrow \xi \in \mathbb{R}$ we may imagine that $x$ could move to $\xi$ only from its right or from its left, that is considering $x > \xi$ or $x < \xi$. This leads to the

**Definition 5.1.6: Unilateral limit**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\xi \in \mathbb{R}$ such that $\xi \in \mathrm{Acc}(D \cap [\xi, +\infty[)$. We say that

$$\exists \lim_{x\to\xi+} f(x) = \ell \in \mathbb{R} \cup \{\pm\infty\},$$

if

$$\forall\ (x_n) \subset D\backslash\{\xi\},\ x_n \longrightarrow \xi+ \implies f(x_n) \longrightarrow \ell. \tag{5.1.4}$$

Similarly, Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\xi \in \mathbb{R}$ such that $\xi \in \mathrm{Acc}(-\infty \cap D)$. We say that

$$\exists \lim_{x\to\xi-} f(x) = \ell \in \mathbb{R} \cup \{\pm\infty\},$$

if

$$\forall\ (x_n) \subset D\backslash\{\xi\},\ x_n \longrightarrow \xi- \implies f(x_n) \longrightarrow \ell. \tag{5.1.5}$$

A similar definition holds for

$$\lim_{x\to\xi-} f(x).$$

Example 5.1.7.

$$\lim_{x\to 0+} \frac{1}{x} = +\infty, \quad \lim_{x\to 0-} \frac{1}{x} = -\infty.$$

Sol. — We check the first, the second being similar. Let $(x_n) \subset D\backslash\{0\}$ be any sequence such that $x_n \longrightarrow 0+$: then

$$f(x_n) = \frac{1}{x_n} \stackrel{\frac{1}{0+}}{\longrightarrow} +\infty. \quad \square$$

If both left/right limits exist, then they should coincide in order the "full" limit exists: this is the content of the

**Proposition 5.1.8**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $x_0 \in \mathrm{Acc}(D \cap [x_0, +\infty[) \bigcap \mathrm{Acc}(D\cap ]-\infty, x_0])$. Then

$$\exists \lim_{x\to x_0} f(x) = \ell, \iff \exists \lim_{x\to x_0+} f(x) = \ell, \ \exists \lim_{x\to x_0-} f(x) = \ell.$$

Proposition 5.1 provides a simple non-existence test:

**Proposition 5.1.9**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\xi \in \mathrm{Acc}(D)$. If

$$\exists\, (x_n), (y_n) \subset D\backslash\{\xi\} \ : \ x_n, y_n \longrightarrow \xi, \text{ and } f(x_n) \longrightarrow \ell_1, \ f(y_n) \longrightarrow \ell_2, \text{ with } \ell_1 \neq \ell_2,$$

then $\lim_{x\to x_0} f(x)$ does not exist.

Example 5.1.10.

$$\nexists \lim_{x\to+\infty} \sin x, \ \nexists \lim_{x\to+\infty} \cos x.$$

Sol. — Take $x_n = 2n\pi$ and $y_n = \frac{\pi}{2} + 2n\pi$. Clearly $x_n, y_n \longrightarrow +\infty$ but

$$\sin x_n = \sin(2n\pi) = 0 \longrightarrow 0, \quad \sin y_n = \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1 \longrightarrow 1.$$

Similarly for cos. $\square$

Example 5.1.11.

$$\nexists \lim_{x\to 0+} \sin\frac{1}{x}, \ \nexists \lim_{x\to 0+} \cos\frac{1}{x}.$$

Sol. — Let's see for cosine: take $x_n$ in such a way that $\frac{1}{x_n} = 2n\pi$, that is $x_n = \frac{1}{2n\pi} \longrightarrow 0+$, and $\frac{1}{y_n} = \frac{\pi}{2} + 2n\pi$, that is $y_n := \frac{1}{\frac{\pi}{2}+2n\pi} \longrightarrow 0+$. Then

$$\cos\frac{1}{x_n} = \cos(2n\pi) = 1 \longrightarrow 1, \quad \cos\frac{1}{y_n} = \cos\left(\frac{\pi}{2} + 2n\pi\right) = 0 \longrightarrow 0. \quad \square$$

## 5.2. Definition of Continuous Function

Operation of limit is of paramount relevance in the entire Mathematics. Many concepts or problems can be properly stated by using this fundamental concept. The first remarkable application of the concept of limit is to the rigorous definition of *continuous function*. A naive definition, usually given to students, is the following: *a continuous function is a function for which we could plot its graph without lifting the pencil from the sheet*. This is suggestive but, unfortunately, not correct. The right definition is

**Definition 5.2.1**

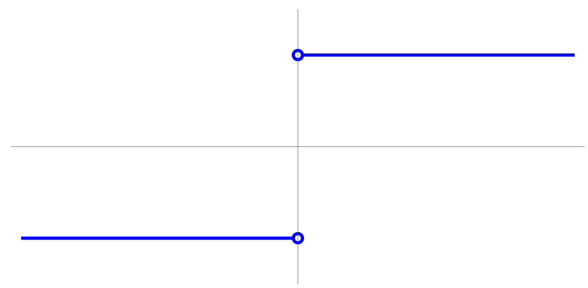
Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\xi \in D \cap \mathrm{Acc}(D)$. We say that $f$ **is continuous at point** $\xi$ if

$$\lim_{x \to \xi} f(x) = f(\xi).$$

If $S$ is a subset of $D$ and $f$ is continuous at every $\xi \in S$, we say that $f$ *is continuous on* $S$ and we write $f \in \mathscr{C}(S)$. Namely, $\mathscr{C}(S)$ denotes the family of functions that are continuous on $S$.

Example 5.2.2. *Let* $f : \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} +1, & x > 0, \\ -1, & x < 0, \end{cases} \equiv \operatorname{sgn} x.$$



*Then* $f \in \mathscr{C}(\mathbb{R}\backslash\{0\})$.

Sol. — Pick $\xi \in D = \mathbb{R}\backslash\{0\}$, for example $\xi > 0$. If $(x_n) \subset D\backslash\{\xi\}$ is such that $x_n \longrightarrow \xi$, then, according to permanence of sign, one has $x_n > 0$, definitely. Thus $f(x_n) \equiv 1$ definitely, therefore $f(x_n) \longrightarrow 1 = f(\xi)$. Similar argument for $\xi < 0$. This proves that $f \in \mathscr{C}(\mathbb{R}\backslash\{0\})$, which is precisely the domain of $f$. Notice that to wonder if $f$ is continuous at $\xi = 0$ is a nonsense because $f$ is not even defined at $\xi = 0$. □

Previous example shows something true in general for elementary functions:

Theorem 5.2.3. *Elementary functions (power, exponential, logarithm, trigonometric, and hyperbolic functions with their inverses) are continuous on their natural domain.*

We will not prove here this important result (we will give some proof later). We limit to check as an exercise the continuity of the square root:

Example 5.2.4. $\sqrt{x}$ *is continuous on* $[0, +\infty[$.

Sol. — Let $\xi > 0$. We have to check that

$$\lim_{x \to \xi} \sqrt{x} = \sqrt{\xi}.$$

Let $(x_n) \subset [0, +\infty[\backslash\{\xi\}$, $x_n \longrightarrow \xi$. Goal: prove that $\sqrt{x_n} \longrightarrow \sqrt{\xi}$. To this aim, notice that

$$\sqrt{x_n} - \sqrt{\xi} = \left(\sqrt{x_n} - \sqrt{\xi}\right) \frac{\sqrt{x_n} + \sqrt{\xi}}{\sqrt{x_n} + \sqrt{\xi}} = \frac{x_n - \xi}{\sqrt{x_n} + \sqrt{\xi}},$$

thus

$$|\sqrt{x_n} - \sqrt{\xi}| = \frac{|x_n - \xi|}{\sqrt{x_n} + \sqrt{\xi}} \leqslant \frac{1}{\sqrt{\xi}} |x_n - \xi|.$$

Since $x_n \longrightarrow \xi$, $|x_n - \xi| \longrightarrow 0$, whence $|\sqrt{x_n} - \sqrt{\xi}| \longrightarrow 0$ by the two policemen theorem or, equivalently, $\sqrt{x_n} \longrightarrow \sqrt{\xi}$ as promised.

In the case $\xi = 0$ the previous argument does not apply. However, we may proceed directly in the following way: let $x_n \longrightarrow 0+$ (of course this is the unique case since $x_n > 0$). We claim that$\sqrt{x_n} \longrightarrow 0$ that is, according to the definition of limit for sequences,

$$\forall \varepsilon > 0, \ \exists N \ : \ \sqrt{x_n} \leqslant \varepsilon, \ \forall n \geqslant N.$$

However, since

$$\sqrt{x_n} \leqslant \varepsilon, \iff 0 \leqslant x_n \leqslant \varepsilon^2.$$

This is true because $x_n \longrightarrow 0$, the conclusion follows. □

By using unilateral limits, we may define **continuity from the left/right at** $\xi$ as

$$\lim_{x \to \xi-} f(x) = f(\xi), \quad \lim_{x \to \xi+} f(x) = f(\xi).$$

Of course: *$f$ is continuous at $\xi$ iff it is continuous from the right and from the left at $\xi$.*

In general, continuity is a property of a function $f$ at some point $\xi$. Continuity at some point $\xi$ does not necessarily extend to other points:

Example 5.2.5. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined as*

$$f(x) = \begin{cases} x, & x \in \mathbb{Q}, \\ \\ -x, & x \notin \mathbb{Q}. \end{cases}$$

*Then $f$ is continuous only at $\xi = 0$.*

Sol. — We first prove $f$ is continuous at $\xi = 0$ that is

$$\lim_{x \to 0} f(x) = f(0) = 0.$$

Let $(x_n) \subset \mathbb{R}\backslash\{0\}$, $x_n \longrightarrow 0$. Of course, $n$ by $n$, $x_n$ is rational or irrational, but we cannot give a general rule when this happens. Nonetheless, we could notice that since

$$-|x| \leqslant f(x) \leqslant |x|, \implies -|x_n| \leqslant f(x_n) \leqslant |x_n|,$$

and because $x_n \longrightarrow 0$ implies $|x_n| \longrightarrow 0$, by two policemen theorem $f(x_n) \longrightarrow 0$ as well. Being $(x_n)$ arbitrary, we conclude.

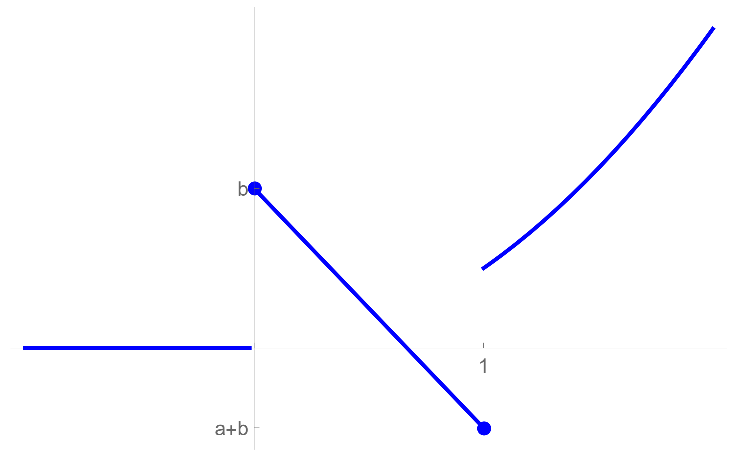
Second, we prove that $f$ **cannot be continuous at any** $\xi \neq 0$. This is a bit more subtle and requires to use *density of rational and irrational numbers in* $\mathbb{R}$ (Theorems 2.5 and 2.5). Pick $\xi \neq 0$: according to density there exists $(x_n) \subset \mathbb{Q}\backslash\{\xi\}$ and $(y_n) \subset \mathbb{Q}^c\backslash\{\xi\}$ such that $x_n, y_n \longrightarrow \xi$. Then,

$$f(x_n) = x_n \longrightarrow \xi, \ \text{ while } \ f(y_n) = -y_n \longrightarrow -\xi,$$

and because $\xi \neq 0$ this means that $\lim_{x \to \xi} f(x)$ does not exist (because we have two different limits for the sequences $f(x_n)$ and $f(y_n)$) . □

Example 5.2.6. *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *defined as*

$$f(x) := \begin{cases} 0, & x < 0, \\ ax + b & 0 \leqslant x \leqslant 1, \\ x^2, & x > 1. \end{cases}$$




*Are there* $a, b \in \mathbb{R}$ *such that* $f \in \mathscr{C}(\mathbb{R})$*?*

Sol. — It is easy to check that $f \in \mathscr{C}(\mathbb{R}\backslash\{0, 1\})$. Thus, the unique question is continuity at $\xi = 0, 1$. We have to check whether or not $\lim_{x\to 0} f(x) = f(0) = b$ and $\lim_{x\to 1} f(x) = f(1) = a + b$. Since $f$ takes different forms at the left/right of these two points, it is better to compute unilateral limits: we have

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} 0 = 0, \quad \lim_{x\to 0+} f(x) = \lim_{x\to 0+} (ax + b) = b.$$

Thus we have that $\exists \lim_{x\to 0} f(x)$ iff $0 = a$. In this case, $\lim_{x\to 0} f(x) = 0 = f(0)$ thus $f$ is also continuous at 0. Similarly

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} (ax + b) = a + b, \quad \lim_{x\to 1+} f(x) = \lim_{x\to 1+} x^2 = 1,$$

thus $\exists \lim_{x\to 1} f(x)$ iff $a + b = 1$ and, in that case, being $f(1) = a + b$ $f$ turns out to be continuous at 1. In conclusion, $f$ is continuous at both 0 and 1 iff

$$\begin{cases} a = 0, \\ \\ a + b = 1, \end{cases} \iff \begin{cases} a = 0, \\ \\ b = 1. \end{cases} \qquad \square$$

Sometimes, a function is defined around a certain accumulation point $\xi$ for $D$ and one wonder if it is possible to extend the definition of $f$ also at point $\xi$ in such a way the extension be continuous at $\xi$. We may formalize this question in the following way. Define

$$\widetilde{f} : D \cup \{\xi\} \longrightarrow \mathbb{R}, \quad \widetilde{f}(x) := \begin{cases} f(x), & x \in D, \\ \\ \ell, & x = \xi. \end{cases}$$

A typical example is the function

$$f : \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}, \quad f(x) := \frac{\sin x}{x}, \ x \neq 0.$$

It is clear that we may give to $\widetilde{f}$ any value at $\xi$, but the question is: *is there a specific $\ell$ such that $\widetilde{f}$ be continuous at point $\xi$?*

Now, in order $\widetilde{f}$ be continuous at $\xi$ we need

$$\lim_{x\to\xi} \widetilde{f}(x) = \widetilde{f}(\xi) = \ell.$$

Since $\widetilde{f}(x) = f(x)$ for $x \neq \xi$ and because the value of $\widetilde{f}$ at point $\xi$ does not enter into the calculation of the limit, we may say that

$$\lim_{x\to\xi} \widetilde{f}(x) = \lim_{x\to\xi} f(x),$$

thus $\widetilde{f}$ *is continuous at* $\xi$ *iff*

$$\boxed{\ell = \lim_{x\to\xi} f(x).}$$

## 5.3. Basic properties of limits

Limit of a function fulfils similar properties of the limit for sequences. This is clear since the definition of the former relies on the second. For example:

**Proposition 5.3.1**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\xi \in \mathrm{Acc}(D)$. If $\lim_{x\to\xi} f(x)$ exists, it is also unique.

Next property is the *permanence of sign*. Roughly speaking, this says that a sequence definitely shares the same sign of its limit. In the language of sequences, definitely means for $n \geqslant N$, naively we would say ”for $n$ big” or with a suggestive language ”for $n$ close to $+\infty$”. Exporting this idea to functions, we should expect something like $f(x)$ has the same sign of $\lim\limits_{x\to\xi} f(x)$ provided $x$ is sufficiently close to $\xi$. We need to make rigorous this concept of ”close to”. To this aim, we need the important

**Definition 5.3.2: neighbourhood**

Let $\xi \in \mathbb{R} \cup \{\pm\infty\}$. We call **neighbourhood of** $\xi$ a set of type

- $I_\xi := [\xi - r, \xi + r]$ with $r > 0$, in the case $\xi \in \mathbb{R}$;
- $I_{+\infty} := [a, +\infty[$, with $a \in \mathbb{R}$, in the case $\xi = +\infty$;
- $I_{-\infty} :=] - \infty, b]$, with $b \in \mathbb{R}$, in the case $\xi = -\infty$.

We have

**Proposition 5.3.3**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\xi \in \mathrm{Acc}(D)$ such that

$$\exists \lim_{x \to \xi} f(x) =: \ell \in \mathbb{R} \cup \{\pm\infty\}.$$

Then

- if $\ell > 0$ [resp. $\ell < 0$], there exists a neighbourhood $I_\xi$ such that $f(x) > 0$ [resp. $f(x) < 0$] for all $x \in D \cap I_\xi \backslash \{\xi\}$;
- if there exists a neighbourhood $I_\xi$ such that $f(x) \geqslant 0$ for all $x \in D \cap I_\xi \backslash \{\xi\}$ then $\ell \geqslant 0$.

Proof. We prove the case $\xi \in \mathbb{R}$ (leaving $\xi = \pm\infty$ as exercise).

**First statement.** Suppose it is false: then

$$\forall\, I_\xi,\ \exists x \in D \cap I_\xi \backslash \{\xi\}\ :\ f(x) \leqslant 0.$$

Taking $I_\xi = \left[\xi - \frac{1}{n}, \xi + \frac{1}{n}\right]$, we can write

$$\forall n \in \mathbb{N},\ \exists x_n \in D \cap \left[\xi - \frac{1}{n}, \xi + \frac{1}{n}\right] \backslash \{\xi\},\ :\ f(x_n) \leqslant 0.$$

Thus, $(x_n) \subset D \backslash \{\xi\}$ and $x_n \longrightarrow \xi$ (because $|x_n - \xi| \leqslant \frac{1}{n}$), therefore by (5.1.2) $f(x_n) \longrightarrow \ell > 0$. Because of the permanence of sign for sequences, we should have $f(x_n) > 0$ definitely. However, $f(x_n) \leqslant 0$ for every $n$, and this is a contradiction.
**Second statement.** Once again let us prove it by contradiction. Suppose $\lim_{x \to \xi} f(x) < 0$. Then, by the first statement, there exists $J_\xi$ such that

$$f(x) < 0,\ \forall x \in D \cap J_\xi \backslash \{\xi\}.$$

But then, taking $x \in I_\xi \cap J_\xi$ (non empty: why?) we would have $f(x) > 0$ and $f(x) < 0$, which is impossible! $\square$

Remark 5.3.4. As already pointed out for sequences, we cannot say that

$$\lim_{x \to \xi} f(x) \geqslant 0, \iff f(x) \geqslant 0,\ \forall x \in I_\xi \backslash \{\xi\},$$

(take, for instance, $f(x) := x \sin \frac{1}{x}$, easily $\lim_{x \to 0} f(x) = 0$ (by the two policemen theorem) but there's no neighbourhood of 0 where $f$ is positive/negative). Neither,

$$\lim_{x \to \xi} f(x) > 0, \iff f(x) > 0,\ \forall x \in I_\xi \backslash \{\xi\},$$

(take $f(x) = x^2$, here $f(x) > 0$ for every $x \in \mathbb{R} \backslash \{0\}$, thus whatever is a neighbourhood $I_0$ we have $f > 0$ on $I_0 \backslash \{0\}$, but $\lim_{x \to 0} f(x) = 0$). $\square$

Following the same kind of ideas, we easily obtain the version of the two policemen argument in the case of limits for functions:

**Theorem 5.3.5: police theorem**

Let $f, g, h : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\xi \in \mathrm{Acc}(D)$. Suppose that

i) there exists $I_\xi$ such that $f(x) \leqslant g(x) \leqslant h(x)$ $\forall x \in D \cap I_\xi \backslash \{\xi\}$;

ii) $\lim_{x\to\xi} f(x) = \lim_{x\to\xi} h(x) = \ell$.

Then $\exists \lim\limits_{x\to\xi} g(x) = \ell$.

Example 5.3.6. *Show that*

$$\lim_{x\to+\infty} \frac{\sin x}{x} = 0.$$

Sol. — We have $-1 \leqslant \sin x \leqslant 1$ for any $x$, therefore

$$-\frac{1}{x} \leqslant \frac{\sin x}{x} \leqslant \frac{1}{x}, \ \forall x \in ]0, +\infty[= I_{+\infty}.$$

Now: $-\frac{1}{x}$ and $\frac{1}{x}$ are two policemen going to 0 as $x \longrightarrow +\infty$ (evident). Therefore $\frac{\sin x}{x} \longrightarrow 0$. □

An important application of the police theorem is the bounded × infinitesimal = infinitesimal rule. We need first to introduce a couple of definitions:

**Definition 5.3.7**

We say that $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ is **bounded on** $S \subset D$ if

$$\exists\, M \geqslant 0, : \ |f(x)| \leqslant M, \ \forall x \in S.$$

**Definition 5.3.8**

We say that $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, is **infinitesimal at** $\xi \in \mathrm{Acc}(D)$ if

$$\lim_{x\to\xi} f(x) = 0.$$

We have

**Corollary 5.3.9**

Let $f, g : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\xi \in \mathrm{Acc}(D)$, such that

i) $f$ be bounded in some $I_\xi \backslash \{\xi\}$;

ii) $g$ be infinitesimal at $\xi$.

Then $f \cdot g$ is infinitesimal at $\xi$.

Example 5.3.10. *Show that*

$$\lim_{x\to 0+} x \sin\frac{1}{x} = 0.$$

Sol. — Let $f(x) := \sin\frac{1}{x}$, defined on $\mathbb{R}\backslash\{0\}$ and $g(x) := x$ defined on $\mathbb{R}$, therefore both defined on $D := \mathbb{R}\backslash\{0\}$. Clearly $0 \in \mathrm{Acc}(D)$, $f$ is bounded on $D$ (hence in any $I_0\backslash\{0\}$) and $g$ is null at 0. Therefore, $f \cdot g$ is null at 0, and this gives the conclusion. □

Finally, as for sequences, monotone functions have always a unilateral limit.

**Theorem 5.3.11**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing function. Then

- if $\xi \in \mathrm{Acc}(D\cap ]-\infty, \xi])$ then
$$\exists \lim_{x\to\xi-} f(x) = \sup\{f(y) \ : \ y \in D, \ y < \xi\}.$$
- if $\xi \in \mathrm{Acc}(D \cap [\xi, +\infty[)$ then
$$\exists \lim_{x\to\xi+} f(x) = \inf\{f(y) \ : \ y \in D, \ y > \xi\}.$$

Of course, a dual statement holds for decreasing functions.

## 5.4. Rules of calculus

Rules of calculus for limits of functions work exactly in the same manner of the rules for limits of sequences.

**Proposition 5.4.1**

Let $f, g : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\xi \in \mathrm{Acc}(D)$ be such that
$$\lim_{x\to\xi} f(x) = \ell_1 \in \mathbb{R}, \quad \lim_{x\to\xi} g(x) = \ell_2 \in \mathbb{R}.$$
Then

i) $\exists \lim_{x\to\xi} (f(x) \pm g(x)) = \ell_1 \pm \ell_2$.
ii) $\exists \lim_{x\to\xi} f(x)g(x) = \ell_1\ell_2$.
iii) if $\ell_2 \neq 0$, $\exists \lim_{x\to\xi} \frac{f(x)}{g(x)} = \frac{\ell_1}{\ell_2}$.

In some cases it is possible to have rules also for infinite limits, as for sequences. Just like in that case, we will use directly the short notations to present the rules:

$$(\pm\infty) + \ell = \pm\infty, \quad (+\infty) + (+\infty) = +\infty, \quad (-\infty) + (-\infty) = -\infty,$$

$$(+\infty) \cdot \ell = \mathrm{sgn}(\ell)\infty, \ (\ell \neq 0), \quad (-\infty) \cdot \ell = -\mathrm{sgn}(\ell)\infty, \ (\ell \neq 0),$$

$$(+\infty) \cdot (+\infty) = +\infty, \quad (+\infty) \cdot (-\infty) = -\infty, \quad (-\infty) \cdot (-\infty) = +\infty,$$

$$\frac{\ell}{\pm\infty} = 0, \ (\ell \in \mathbb{R}), \quad \frac{+\infty}{\ell} = \mathrm{sgn}(\ell)\infty, \ (\ell \neq 0), \quad \frac{-\infty}{\ell} = -\mathrm{sgn}(\ell)\infty, \ (\ell \neq 0).$$

**Definition 5.4.2**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\xi \in \mathrm{Acc}(D)$. We say that $\lim_{x\to\xi} f(x) = \ell+$ (here, of course, $\ell \in \mathbb{R}$) if

i) $\lim_{x\to\xi} f(x) = \ell$;

ii) there exists $I_\xi$ such that $f > \ell$ on $I_\xi \backslash \{\xi\}$.

With this definition we have also

$$\frac{+\infty}{0+} = \frac{-\infty}{0-} = +\infty, \quad \frac{+\infty}{0-} = \frac{-\infty}{0+} = -\infty.$$

The following are **indeterminate forms**:

$$(\pm\infty) + (\mp\infty), \text{ (opposite signs)}, \quad (\pm\infty)\cdot 0, \quad \frac{0}{0}, \ \frac{\pm\infty}{\pm\infty}.$$

In addition, we also have the following indeterminate forms for the powers

$$1^{\pm\infty}, \quad (0+)^0, \quad (0+)^{+\infty},$$

that are reduced to $0\cdot\infty$ through

$$f(x)^{g(x)} = e^{g(x)\log f(x)}.$$

Example 5.4.3. *Compute*

$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}.$$

Sol. — As $x \longrightarrow 0$ clearly $1+x, 1-x \longrightarrow 1$ and because of continuity of the square root, $\sqrt{1+x} \longrightarrow \sqrt{1} = 1$. Similarly $\sqrt{1-x} \longrightarrow \sqrt{1} = 1$, so that we have a form $\frac{0}{0}$. If one of the terms in the numerator were 'prevailing' –namely, if $\frac{\sqrt{1+x}}{\sqrt{1-x}}$ or its inverse would tend to zero— we might factorize the numerator with it. Yet

$$\frac{\sqrt{1+x}}{\sqrt{1-x}} = \sqrt{\frac{1+x}{1-x}} \longrightarrow \sqrt{\frac{1}{1}} = \sqrt{1}, \ x \longrightarrow 0.$$

So, we have to find another way to find the limit. Using the identity $(a+b)(a-b) = a^2 - b^2$, we get

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \frac{\sqrt{1+x}-\sqrt{1-x}}{x}\,\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}} = \frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$= \frac{2}{\sqrt{1+x}+\sqrt{1-x}} \longrightarrow \frac{2}{\sqrt{1}+\sqrt{1}} = 1. \quad \square$$

An important tool available in the limits of functions is *change of variables*. Suppose that, computing

$$\lim_{x\to x_0} f(x), \tag{5.4.1}$$

we see that $f(x) = g(h(x))$. Hence, it would be natural to set $y = h(x)$ and if $y = h(x) \longrightarrow y_0$ we may expect that

$$\lim_{x\to x_0} f(x) \stackrel{?}{=} \lim_{y\to y_0} g(y). \tag{5.4.2}$$

Hopefully, the last limit is easier than the first one. We wonder if (5.4.2) is correct. The answer is yes provided $h(x) \neq x_0$ when $x \longrightarrow x_0$. Precisely and a function $g : E \to \mathbb{R}$ such that

**Proposition 5.4.4: Change of variable**

Assume that $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $x_0 \in \mathrm{Acc}D$. Suppose that there exists a neighbourhood $I_{x_0}$ of $x_0$ functions $h : D \longrightarrow \mathbb{R}$ $g : h(D) \to \mathbb{R}$ such that such that

i) $f(x) = g(h(x))$, $\forall x \in I_{x_0} \backslash \{x_0\}$;

ii) $\lim_{x\to x_0} h(x) = y_0$;

iii) $h(x) \neq y_0$ for all $x \in I_{x_0} \backslash \{x_0\}$.

If $y_0 \in \mathrm{Acc}(h(D))$ and

$$\exists \lim_{y\to y_0} g(y) = \ell \in \mathbb{R} \cup \{\pm\infty\},$$

then

$$\Longrightarrow \quad \exists \lim_{x\to x_0} f(x) = \ell \in \mathbb{R} \cup \{\pm\infty\}.$$

Proof. We use the definition: let $(x_n) \subset D\backslash\{x_0\}$, $x_n \longrightarrow x_0$. We may assume $x_n \in I_{x_0} \backslash \{x_0\}$: thus, by i)

$$f(x_n) = g(h(x_n)) =: g(y_n), \text{ where } y_n = h(x_n).$$

By iii) one as $y_n \neq y_0$, and by ii) it is $y_n = h(x_n) \longrightarrow y_0$. Thus, since

$$\lim_{y\to y_0} g(y) = \ell, \quad \Longrightarrow \quad f(x_n) = g(y_n) \longrightarrow \ell.$$

This being true no matter who $(x_n)$ is, according to (5.1.2) we get the conclusion. □

Remark 5.4.5. *Condition iii) might look not natural. In fact, it is essential for the validity of the thesis. Consider for example the case when $h(x) \equiv 0$, and $g(y) = |y|$ $\forall y \neq 0$, and $g(0) = 1$. Clearly, $\lim_{y\to 0} g(y) = 0$. Instead, setting $f := g \circ h$ one has $f(x) = g(h(x)) = g(0) = 1$ for every $x$, so that $\lim_{x\to 0} f(x) = 1$. Notice that $g$ is not continuous at $y_0 = 0$ (i.e. its limit at $0$ is different from the value $g(0)$). As a matter of fact, in Proposition 6.1 we will see that one can drop hypothesis iii) as soon as $g$ is continuous at $y_0$.*

The practical use of this technique is much easier than what it appears.

Example 5.4.6. *Compute*

$$\lim_{x\to+\infty} \frac{e^{\sqrt{\log x}}}{\sqrt{x}}.$$

Sol. — Clearly $\log x \longrightarrow +\infty$, so $\sqrt{\log x} \longrightarrow +\infty$ hence $e^{\sqrt{\log x}} \longrightarrow +\infty$. Since $\sqrt{x} \longrightarrow +\infty$ as well, we have an indeterminate form $\frac{\infty}{\infty}$ [1]. Let us change variable setting

$$\frac{e^{\sqrt{\log x}}}{\sqrt{x}} \overset{y:=\sqrt{\log x}\longrightarrow+\infty,\ y^2=\log x,\ x=e^{y^2}}{=} \frac{e^y}{\sqrt{e^{y^2}}} = \frac{e^y}{e^{\frac{y^2}{2}}} = e^{y-\frac{y^2}{2}}.$$

[1]Be careful: $e^{\sqrt{\log x}} \neq \sqrt{x}$ (someone could think that $e^{\sqrt{\log x}} = \sqrt{e^{\log x}} = \sqrt{x}$,which is wrong!!)

Because $y - \frac{y^2}{2} = -\frac{y^2}{2}\left(1 - \frac{2}{y}\right) \overset{+\infty\cdot 1}{\longrightarrow} -\infty$,

$$\lim_{x\to+\infty} \frac{e^{\sqrt{\log x}}}{\sqrt{x}} \overset{y=\sqrt{\log x}\longrightarrow+\infty}{=} \lim_{y\to+\infty} e^{y-\frac{y^2}{2}} \longrightarrow e^{-\infty} = 0. \qquad \square$$

## 5.5. Fundamental limits

To compute limits, the main problems are indeterminate forms. These are generated by a competition between different terms. It is therefore fundamental to be able to compare different quantities to determine the dominant ones. As for sequences, the comparison is made through ratios. We start with the

**Definition 5.5.1**

Let $f(x)$, $g(x)$ two infinities at $\xi \in \mathbb{R} \cup \{\pm\infty\}$, that is

$$\lim_{x\to\xi} f(x) = \pm\infty, \quad \lim_{x\to\xi} g(x) = \pm\infty.$$

We say that $f(x)$ **is infinite of greater order than** $g(x)$ **at** $\xi$ (notation: $f(x) \gg_\xi g(x)$) if

$$\lim_{x\to\xi} \frac{g(x)}{f(x)} = 0, \text{ or, equivalently, } \lim_{x\to\xi} \frac{|f(x)|}{|g(x)|} = +\infty$$

Exercise 5.5.2. *Show that* $\lim_{x\to\xi} \frac{g(x)}{f(x)} = 0$ *is actually equivalent* $\lim_{x\to\xi} \frac{|f(x)|}{|g(x)|} = +\infty$

Therefore, for example, when we have an indeterminate form $(+\infty) - (+\infty)$ and one term has greater order then the other(s), we may solve the indeterminacy factorizing the dominant term:

$$f(x) - g(x) = f(x)\left(1 - \frac{g(x)}{f(x)}\right) = f(x) \cdot k(x),$$

where $k(x) \longrightarrow 1$, that is, we can transform $(+\infty) - (+\infty)$ into $(+\infty) \cdot 1$, which is not an indeterminate form.

Basic examples of infinities at $+\infty$ are $x^\alpha$ $(\alpha > 0)$, $a^x$ $(a > 1)$ and $\log_b x$ $(b > 1)$. It is easy to check that

$$x^\alpha \gg_{+\infty} x^\beta, \ \forall \alpha > \beta > 0, \quad a^x \gg_{+\infty} b^x, \ \forall a > b > 1.$$

Indeed

$$\frac{x^\beta}{x^\alpha} = \frac{1}{x^{\alpha-\beta}} \longrightarrow 0, \text{ (because } \alpha - \beta > 0), \quad \frac{b^x}{a^x} = \left(\frac{b}{a}\right)^x \longrightarrow 0, \text{ (because } 0 < \frac{b}{a} < 1).$$

With logarithms, the situation is different because $x = b^{\log_b x} = a^{(\log_b x)(\log_a b)}$ that is $\log_a x = (\log_b a)(\log_b x)$, so, in particular, $\frac{\log_b x}{\log_a x} \equiv \log_b a$. More difficult is the mixed comparison. Similarly to sequences, we have the

**Proposition 5.5.3**

$$a^x \gg_{+\infty} x^\alpha \gg_{+\infty} \log_b x, \quad \forall a > 1, \ \alpha > 0, \ b > 1.$$

Example 5.5.4. *Compute*

$$\lim_{x\to+\infty}\left(\frac{1}{\log x}\right)^{\frac{\log x}{x}}.$$

Sol. — Being $x \gg_{+\infty} \log x$, $\frac{\log x}{x} \longrightarrow 0$, and $\frac{1}{\log x} \longrightarrow 0+$, we have the indeterminate form $(0+)^0$. Notice that

$$\left(\frac{1}{\log x}\right)^{\frac{\log x}{x}} = e^{\frac{\log x}{x}\log\frac{1}{\log x}} = e^{-\frac{(\log x)(\log(\log x))}{x}},$$

so we have to compute

$$\lim_{x\to+\infty}\frac{(\log x)(\log\log x)}{x},$$

which is a form $\frac{\infty}{\infty}$. Changing variable,

$$\frac{(\log x)(\log\log x)}{x} \overset{y=\log x\longrightarrow+\infty}{=} \frac{y\log y}{e^y} = \frac{y^2}{e^y}\frac{\log y}{y} \longrightarrow 0,\ \text{because } e^y \gg_{+\infty} y^2,\ y \gg_{+\infty} \log y. \qquad \square$$

Example 5.5.5.

$$\lim_{x\to0+} x^\alpha|\log x|^\beta = 0,\ \forall\alpha,\beta>0.$$

Sol. — The limit presents as a form $0\cdot\infty$. Changing variable,

$$\lim_{x\to0+} x^\alpha|\log x|^\beta \overset{y=\log x\longrightarrow-\infty}{=} \lim_{y\to-\infty} e^{\alpha y}|y|^\beta = \lim_{y\to-\infty}\frac{|y|^\beta}{e^{-\alpha y}} \overset{z=-\alpha y\longrightarrow+\infty}{=} \lim_{z\to+\infty}\frac{\frac{z^\beta}{\alpha^\beta}}{e^z} = \frac{1}{\alpha^\beta}\lim_{z\to+\infty}\frac{z^\beta}{e^z} = 0. \qquad \square$$

Example 5.5.6. *Compute*

$$\lim_{x\to0+}\left(\frac{1}{x}\right)^x.$$

Sol. — We have a form $(+\infty)^0$, and, by continuity of the exponential,

$$\left(\frac{1}{x}\right)^x = e^{x\log\frac{1}{x}} = e^{-x\log x} \longrightarrow e^{-0} = 1. \qquad \square$$

Recall now that

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Replacing $n$ by $x$ easily, this limit may be extended to the real variable. From this, a number of variations can be drawn:

**Proposition 5.5.7**

(5.5.1) $$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e,\quad \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e,\quad \lim_{x\to0}(1+x)^{1/x} = e.$$

In particular

(5.5.2) $$\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^x = e^a,\ \forall a\in\mathbb{R}.$$

Proof. For the first, one has

$$\left(1+\frac{1}{[x]+1}\right)^{[x]+1}\left(1+\frac{1}{[x]+1}\right)^{-1} \leq \left(1+\frac{1}{x}\right)^{x} \leq \left(1+\frac{1}{[x]}\right)^{[x]}\left(1+\frac{1}{[x]}\right)$$

Since

$$\lim_{x\to+\infty}\left(1+\frac{1}{[x]+1}\right)^{[x]+1} \underset{n=[x]+1}{=} \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n} = e,$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]} \underset{n=[x]}{=} \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n} = e,$$

and (still by change of variable $n = [x] + 1$ and $n = [x]$, respectively,

$$\lim_{x\to+\infty}\left(1+\frac{1}{[x]+1}\right)^{-1} = 1 = \lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)$$

Hence, by the Two Policemen Theorem we get

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x} = e,$$

as desired.

For the second

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} \overset{y=-x\longrightarrow+\infty}{=} \lim_{y\to+\infty}\left(1-\frac{1}{y}\right)^{-y} = \lim_{y\to+\infty}\left(\frac{y-1}{y}\right)^{-y} = \lim_{y\to+\infty}\left(\frac{y}{y-1}\right)^{y}$$

$$= \lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right) \overset{z=y-1\longrightarrow+\infty}{=} \lim_{z\to+\infty}\left(1+\frac{1}{z}\right)^{z}\left(1+\frac{1}{z}\right) = e\cdot 1 = e.$$

For the third

$$\lim_{x\to 0+}(1+x)^{1/x} \overset{y=\frac{1}{x}\longrightarrow+\infty}{=} \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y} = e,$$

$$\lim_{x\to 0-}(1+x)^{1/x} \overset{y=\frac{1}{x}\longrightarrow-\infty}{=} \lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^{y} = e,$$

from which the conclusion follows. Finally, if $a > 0$,

$$\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^{x} \overset{y=\frac{x}{a}\longrightarrow+\infty}{=} \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{ay} = \lim_{y\to+\infty}\left(\left(1+\frac{1}{y}\right)^{y}\right)^{a}.$$

Now: $\left(1+\frac{1}{y}\right)^{y} \longrightarrow e$. By continuity of power (see the next Chapter)

$$\left(\left(1+\frac{1}{y}\right)^{y}\right)^{a} \longrightarrow e^{a}. \quad \square$$

Assuming the continuity of power, exponential and logarithm (see the next Chapter), we obtain some essential limits

**Corollary 5.5.8**

$$\lim_{x\to 0}\frac{\log(1+x)}{x}=1,\quad \lim_{x\to 0}\frac{e^x-1}{x}=1,\quad \lim_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\alpha,\ \forall\alpha\neq 0. \tag{5.5.3}$$

PROOF. For the first

$$\lim_{x\to 0}\frac{\log(1+x)}{x}=\lim_{x\to 0}\frac{1}{x}\log(1+x)=\lim_{x\to 0}\log(1+x)^{1/x}=\log e=1.$$

For the second

$$\lim_{x\to 0}\frac{e^x-1}{x}\overset{y=e^x-1\longrightarrow 0}{=}\lim_{y\to 0}\frac{y}{\log(1+y)}=\lim_{y\to 0}\frac{1}{\frac{\log(1+y)}{y}}=1.$$

Finally,

$$\lim_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\lim_{x\to 0}\tfrac{e^{\alpha\log(1+x)}-1}{x}\overset{y=\alpha\log(1+x)\longrightarrow 0}{=}\lim_{y\to 0}\tfrac{e^y-1}{e^{y/\alpha}-1}$$

$$=\lim_{y\to 0}\frac{e^y-1}{y}\frac{y/\alpha}{e^{y/\alpha}-1}\alpha=1\cdot 1\cdot\alpha=\alpha. \qquad\square$$

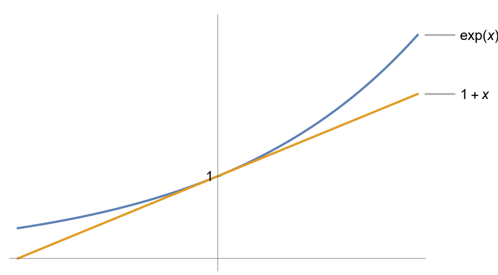
These limits contain some information on the behaviour of some elementary functions when the argument is close to 0. For example, the exponential limit says that

$$\frac{e^x-1}{x}\approx 1,\ \text{when } x\approx 0,\quad\Longleftrightarrow\quad e^x\approx 1+x.$$

This formula delivers two important messages:

- *numerical message*: to compute $e^x$ when $x\approx 0$ we may use a polynomial $1+x$, so for instance $e^{0,1}\approx 1,1$. The point is that this is a way to replace the calculus of $e^x$ (for which we do not have a simple formula) by a computationally simple function: a polynomial. This mechanism will be made general by the Taylor formula.

- *geometrical message*: when $x\approx 0$, graphs of $e^x$ and $1+x$ are very close. Now, $y=1+x$ is a straight line, coinciding with $e^x$ just at $x=0$ (both are equal to 1). The interpretation of the approximation $e^x\approx 1+x$ is that the line $y=1+x$ is *tangent* to the graph of $e^x$ at point $x=0$.




To make more precise these considerations, let us consider the *gap* between $e^x$ and $1+x$, that is $e^x-(1+x)$. We notice that,

$$\frac{e^x-(1+x)}{x}=\frac{e^x-1}{x}-1\longrightarrow 0,$$

thus *the gap is an infinitesimal of greater order than $x$ when $x\longrightarrow 0$*. This gives a precise ground to both previous messages. It is the moment to introduce two important concepts:

**Definition 5.5.9**

Given two functions $f(x), g(x) : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ for which $\xi \in \mathrm{Acc}(D)$, we say that $f(x)$ is **asymptotic** to $g(x)$ at point $\xi$ (notation $f(x) \sim_\xi f(x)$) if

$$\lim_{x\to\xi} \frac{f(x)}{g(x)} = 1.$$

Thus, in particular,

$$e^x - 1 \sim_0 x, \quad \log(1+x) \sim_0 x, \quad (1+x)^\alpha - 1 \sim_0 x.$$

Notice that these three quantities are all asymptotic to the same quantity $x$ at 0 but of course they are entirely different things.

**Definition 5.5.10**

Let $f(x), g(x)$ two infinitesimals at $x_0$, that is

$$\lim_{x\to x_0} f(x) = 0, \quad \lim_{x\to x_0} g(x) = 0.$$

We say that $g(x)$ is **is infinitesimal of greater order than** $f(x)$ **at** $x_0$

$$\lim_{x\to x_0} \frac{g(x)}{f(x)} = 0.$$

Thus, in particular,

$$e^x - (1+x) = o(x), \iff e^x = 1 + x + o(x).$$

Similarly

$$\log(1+x) = x + o(x), \quad (1+x)^\alpha = 1 + \alpha x + o(x).$$

These are the first examples of *Taylor formulas* which will be rigorously introduced and proved in the chapter devoted to derivatives.

We now present some important limits of trigonometric functions. Assuming sin and cos continuous functions, we have

$$\lim_{x\to 0} \sin x = \sin 0 = 0, \quad \lim_{x\to 0}(1-\cos x) = 1 - \cos 0 = 0.$$

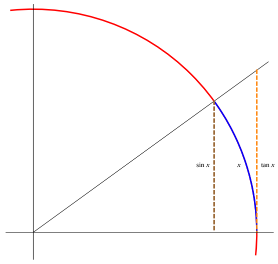
A fundamental couple of limits concern the asymptotic behaviour of these quantities as $x \longrightarrow 0$: we have

**Theorem 5.5.11**

(5.5.4) $$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.$$

In other words, $\sin x \sim_0 x$ and $\cos x \sim_0 1 - \frac{x^2}{2}$.

Proof. The key ingredient of the proof is the remarkable inequality




$$0 \leqslant \sin x \leqslant x \leqslant \tan x, \ \forall x \in \left]0, \frac{\pi}{2}\right[, \tag{5.5.5}$$

which has already proved for a similar resul for sequences. By this and dividing by $\sin x$, for $x \in \left]0, \frac{\pi}{2}\right[$ we have

$$0 \leqslant 1 \leqslant \frac{x}{\sin x} \leqslant \frac{1}{\cos x},$$

that is

$$\cos x \leqslant \frac{\sin x}{x} \leqslant 1, \ \forall x \in \left]0, \frac{\pi}{2}\right[. \tag{5.5.6}$$

When $x \longrightarrow 0+$, by two policemen theorem we have

$$\lim_{x \to 0+} \frac{\sin x}{x} = 1.$$

On the other side

$$\lim_{x \to 0-} \frac{\sin x}{x} \overset{y=-x\longrightarrow 0+}{=} \lim_{y \to 0+} \frac{\sin(-y)}{-y} = \lim_{y \to 0+} \frac{\sin y}{y} = 1,$$

and this proves the first of (5.5.4). About the second we have

$$\frac{1-\cos x}{x^2} = \frac{1-\cos x}{x^2}\frac{1+\cos x}{1+\cos x} = \frac{1-(\cos x)^2}{x^2(1+\cos x)} = \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x} \longrightarrow \frac{1}{2}, \ x \longrightarrow 0. \quad \square$$

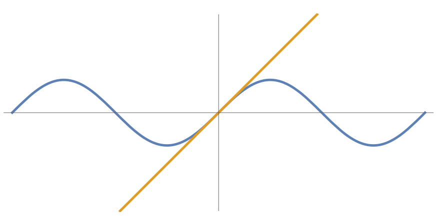
Again, notice that

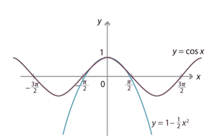
$$\lim_{x \to 0} \frac{\sin x - x}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x} - 1\right) = 0, \implies \sin x - x = o(x), \implies \sin x = x + o(x),$$

and

$$\lim_{x \to 0} \frac{\cos x - \left(1 - \frac{x^2}{2}\right)}{x^2} = \lim_{x \to 0} \left(\frac{\cos x - 1}{x^2} + \frac{1}{2}\right) = 0, \implies \cos x = 1 - \frac{x^2}{2} + o(x^2).$$

Here below, the figure represents the two approximations. Clearly, and this is a good example to point out this, these approximations have this meaning when $x \approx 0$. Far from 0, the sine does not look as a straight line and neither cosine looks like a parabola.





Quadratic approximation to the function cos x.

Example 5.5.12. *Compute*

$$\lim_{x \to 0} \frac{(1-\cos(3x))^2}{x^2(1-\cos(x))}.$$

Sol. — It is easy to recognize a form $\frac{0}{0}$ (use continuity of cos). We manipulate the expression to reduce it to known limits:

$$\frac{(1-\cos(3x))^2}{x^2(1-\cos(x))} = \left[(3x)^2\frac{1-\cos(3x)}{(3x)^2}\right]^2 \frac{1}{x^2}\frac{1}{x^2\frac{1-\cos(x)}{x^2}} = \frac{(3x)^4}{x^4}\left[\frac{1-\cos(3x)}{(3x)^2}\right]^2\left[\frac{1-\cos(x)}{x^2}\right]^{-1}$$

$$= 3^4\left[\frac{1-\cos(3x)}{(3x)^2}\right]^2\left[\frac{1-\cos(x)}{x^2}\right]^{-1}.$$

Now

$$\lim_{x\to 0}\frac{1-\cos(3x)}{(3x)^2} \overset{y=3x\longrightarrow 0}{=} \lim_{y\to 0}\frac{1-\cos y}{y^2} = \frac{1}{2},$$

so

$$\lim_{x\to 0}\frac{(1-\cos(3x))^2}{x^2(1-\cos(x))} = 3^4\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{-1} = \frac{81}{2}. \quad \square$$

Example 5.5.13. *Compute*

$$\lim_{x\to 0}\frac{\log(1+(\sin x)^2)}{1-\cos x}.$$

Sol. — As $x \longrightarrow 0$ $\sin x \longrightarrow 0$, so $1+(\sin x)^2 \longrightarrow 1$, and $\log(1+(\sin x)^2) \longrightarrow \log 1 = 0$ (continuity of logarithm). In addition, the denominator goes to 0, so we have a form $\frac{0}{0}$. The numerator is of type $\log(1+y)$ with $y \longrightarrow 0$. Recalling (5.5.3) and (5.5.4) it's natural to write

$$\frac{\log(1+(\sin x)^2)}{1-\cos x} = \frac{\log(1+(\sin x)^2)}{(\sin x)^2}\frac{(\sin x)^2}{x^2}\frac{x^2}{1-\cos x}.$$

Now

$$\lim_{x\to 0}\frac{\log(1+(\sin x)^2)}{(\sin x)^2} \overset{y=(\sin x)^2\longrightarrow 0}{=} \lim_{y\to 0}\frac{\log(1+y)}{y} = 1,$$

therefore

$$\frac{\log(1+(\sin x)^2)}{1-\cos x} = \frac{\log(1+(\sin x)^2)}{(\sin x)^2}\frac{(\sin x)^2}{x^2}\frac{x^2}{1-\cos x} \longrightarrow 1\cdot 1^2\cdot\frac{1}{1/2} = 2. \quad \square$$

Example 5.5.14. *Compute*

$$\lim_{x\to 0}(\cos x)^{1/x^2}.$$

Sol. — We have a form $1^{+\infty}$. Transforming into an exponential

$$(\cos x)^{1/x^2} = e^{\frac{1}{x^2}\log(\cos x)},$$

so we are reduced to compute

$$\lim_{x\to 0}\frac{\log(\cos x)}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{\log(1+(\cos x-1))}{x^2} = \lim_{x\to 0}\frac{\log(1+(\cos x-1))}{\cos x-1}\frac{\cos x-1}{x^2}.$$

Noticed that

$$\lim_{x\to 0}\frac{\log(1+(\cos x-1))}{\cos x-1} \overset{y=\cos x-1\longrightarrow 0}{=} \lim_{y\to 0}\frac{\log(1+y)}{y} = 1,$$

we have

$$\frac{\log(1+(\cos x-1))}{\cos x-1}\frac{\cos x-1}{x^2} \longrightarrow 1\cdot\left(-\frac{1}{2}\right) = -\frac{1}{2}.$$

By continuity of the exponential we have

$$(\cos x)^{1/x^2} = e^{\frac{1}{x^2}\log(\cos x)} \longrightarrow e^{-1/2}. \quad \square$$

Example 5.5.15. *Compute*

$$\lim_{x\to 0} \frac{\sqrt[5]{1+3x^4}-\sqrt{1-2x}}{\sqrt[3]{1+x}-\sqrt{1+x}}.$$

Sol. — Easily we have o form $\frac{0}{0}$. Writing roots as powers

$$\frac{(1+3x^4)^{1/5}-(1-2x)^{1/2}}{(1+x)^{1/3}-(1+x)^{1/2}} = \frac{\frac{(1+3x^4)^{1/5}-1}{3x^4}3x^4 - \frac{(1-2x)^{1/2}-1}{-2x}(-2x)}{\frac{(1+x)^{1/3}-1}{x}x - \frac{(1+x)^{1/2}-1}{x}x} = \frac{\frac{(1+3x^4)^{1/5}-1}{3x^4}3x^3 - \frac{(1-2x)^{1/2}-1}{-2x}(-2)}{\frac{(1+x)^{1/3}-1}{x} - \frac{(1+x)^{1/2}-1}{x}}$$

Now

$$\lim_{x\to 0}\frac{(1+3x^4)^{1/5}-1}{3x^4} \overset{y=3x^4}{=} \lim_{y\to 0}\frac{(1+y)^{1/5}-1}{y} = \frac{1}{5},$$

and similarly

$$\frac{(1-2x)^{1/2}-1}{-2x} \longrightarrow \frac{1}{2}, \quad \frac{(1+x)^{1/3}-1}{x} \longrightarrow \frac{1}{3}, \quad \frac{(1+x)^{1/2}-1}{x} \longrightarrow \frac{1}{2}.$$

Gluing all togheter

$$\frac{\sqrt[5]{1+3x^4}-\sqrt{1-2x}}{\sqrt[3]{1+x}-\sqrt{1+x}} \longrightarrow \frac{\frac{1}{5}\cdot 0 - \frac{1}{2}(-2)}{\frac{1}{3}-\frac{1}{2}} = 6. \quad \square$$

## 5.6. The little $o$ notation, and the Infinitesimals Substitution Principle (ISP)

**Definition 5.6.1**

Let $h : D \to \mathbb{R}$ be a function, and let $x_0 \in \text{Acc}(D)$ be such that $h(x) \neq 0$ for any $x \in I_{x_0} \cap D$, where $I_{x_0}$ is a suitable a neighbourhood of $x_0$. We say that a function $f : E \to \mathbb{R}$ such that $E \supseteq I_{x_0}$ is **little $o$ of $h$ for $x \longrightarrow x_0$**, and write

$$f = o(h) \quad x \longrightarrow x_0,$$

if

$$\lim_{x\to x_0}\frac{f(x)}{h(x)} = 0.$$

Remark 5.6.2. Notice that if $f, g$ are infinite for $x \longrightarrow x_0$, $f$ is higher order infinite than $g$ if and only if $g = o(f)$ (at $x_0$). Similarly, if $f, g$ are infinitesimal for $x \longrightarrow x_0$ (and there exists a neighbourhood $I_{x_0}$ of $x_0$ such that $g(x) \neq 0\ \forall x \in I_{x_0}\setminus\{x_0\}$), $f$ is higher order infinitesimal than $g$ if and only if $g = o(f)$ (at $x_0$). Finally let us observe that $f = o(1)$ for $x \longrightarrow x_0$ (or, equivalently $f = o(k)$, with $k \in \mathbb{R}\setminus\{0\}$) means exactly that $f$ is infinitesimal for $x \longrightarrow x_0$.

Let us see how $o()$ behaves with operations.

**Proposition 5.6.3**

Let $h$, $k$, and $\ell$ be functions defined in a neighbourhood $I$ of a point $x_0 \in \mathbb{R} \cup \{\pm\infty\}$, and assume

$$h(x) \neq 0, \quad k(x) \neq 0, \quad \ell(x) \neq 0, \quad \forall x \in I \backslash \{x_0\}.$$

Then the following holds true:

i) $o(h) \cdot o(k) = o(hk)$, that is, if $f = o(h)$, $g = o(k)$, then $fg = o(hk)$;

ii) $\ell \cdot o(h) = o(\ell h)$, that is, if $f = o(h)$, then $\ell f = o(\ell f)$;

iii) $o(h + o(h)) = o(h)$, that is, if $w = o(h)$ and $r = o(h + w)$, then $r = o(h)$.

Proof. If $f = o(h)$ and $g = o(k)$ then, by definition, $\frac{f(x)}{h(x)} \to 0$ and $\frac{g(x)}{k(x)} \to 0$, so that $\frac{f(x)g(x)}{h(x)k(x)} \to 0$, so i) is proved. Moreover, $\frac{\ell(x)f(x)}{\ell(x)h(x)} = \frac{f(x)}{h(x)} \to 0$, which proves ii). Finally, if $r = o(h+w)$, with $w = o(h)$, one has

$$\lim_{x\to x_0} \frac{r(x)}{h(x)} = \lim_{x\to x_0} \frac{r(x)}{h(x)+w(x)} \frac{h(x)+w(x)}{h(x)} = \lim_{x\to x_0} \frac{r(x)}{h(x)+w(x)} \lim_{x\to x_0} \frac{h(x)+w(x)}{h(x)} =$$

$$= 0 \cdot \lim_{x\to x_0} \left(1 + \frac{w(x)}{h(x)}\right) = 0 \cdot 1 = 0,$$

so iii) is proved as well.

**Proposition 5.6.4:** *Infinitesimals Substitution Principle (ISP)*

Let $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$ be functions, and let $x_0 \in Acc(D)$, and assume that there exists the limit[a]

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \ell \in \mathbb{R} \cup \{\pm\infty\}.$$

Then

$$\lim_{x\to x_0} \frac{f(x)+o(f)(x)}{g(x)+o(g)(x)} = \lim_{x\to x_0} \frac{f(x)}{g(x)} = \ell.$$

[a] It is implicitly meant that $f(x) \neq 0$ and $g(x) \neq 0$ $\forall x \in I \backslash x_0$, where $I$ is a suitable neighbourhood of $x_0$.

Proof. It is sufficient to factorize with $f(x)$ at the numerator and with $g(x)$ at the denominator:

$$\lim_{x\to x_0} \frac{f(x)+o(f)(x)}{g(x)+o(g)(x)} = \lim_{x\to x_0} \frac{f(x)\left(1+\frac{o(f)(x)}{f(x)}\right)}{g(x)\left(1+\frac{o(g)(x)}{g(x)}\right)} = \lim_{x\to x_0} \frac{f(x)}{g(x)} \cdot \lim_{x\to x_0} \frac{1+\frac{o(f)(x)}{f(x)}}{1+\frac{o(g)(x)}{g(x)}} = \ell \cdot 1 = \ell$$

□

Example 5.6.5. Let us solve two of the previous examples by making use of the ISP:

$$\lim_{x\to 0} \frac{\log(1+(\sin x)^2)}{1-\cos x} = \lim_{x\to 0} \frac{(\sin x)^2 + o((\sin x)^2)}{\frac{x^2}{2} + o(x^2)} = \lim_{x\to 0} \frac{(\sin x)^2 + o((\sin x)^2)}{\frac{x^2}{2} + o\left(\frac{x^2}{2}\right)} \overset{ISP}{=} \lim_{x\to 0} \frac{(\sin x)^2}{\frac{x^2}{2}} = 2\cdot 1 = 2$$

$$\lim_{x\to 0}\frac{\sqrt[5]{1+3x^4}-\sqrt{1-2x}}{\sqrt[3]{1+x}-\sqrt{1+x}} = \lim_{x\to 0}\frac{(1+3x^4)^{1/5}-(1-2x)^{1/2}}{(1+x)^{1/3}-(1+x)^{1/2}} = \lim_{x\to 0}\frac{1+\frac{1}{5}3x^4+o(3x^4)-1-\frac{1}{2}2x+o(2x)}{1+\frac{1}{3}x+o(x)-1-\frac{1}{2}x)+o(x)} =$$

$$\lim_{x\to 0} = \frac{-x+o(x)}{-\frac{x}{6}+o(x)} \overset{ISP}{=} \lim_{x\to 0} = \frac{-x}{-\frac{x}{6}} = -6$$

### 5.7. Hyperbolic functions

It is time to introduce two new functions:

$$\sinh x := \frac{e^x - e^{-x}}{2},\ x\in\mathbb{R},\ \text{(hyperbolic sine)},\quad \cosh x := \frac{e^x + e^{-x}}{2},\ x\in\mathbb{R},\ \text{(hyperbolic cosine)}.$$

Despite they do not seem to be relatives of sin and cos, there are several analogies that suggest the contrary. Actually, it might be seen that, as functions of complex variables, hyperbolic sine and trigonometric sine are closely related. The origin of the name (hyperbolic functions) comes from a remarkable property:

(5.7.1) $$(\cosh x)^2 - (\sinh x)^2 = 1,\ \forall x\in\mathbb{R}.$$

This means that the point $(\cosh x, \sinh x)$ belongs to the hyperbola $\xi^2 - \eta^2 = 1$ in the plane $\xi\eta$. The first important properties of sinh and cosh are contained in the

**Proposition 5.7.1**

It holds

i) $\cosh 0 = 1$, $\sinh 0 = 1$.
ii) cosh is even, sinh is odd.
iii) cosh and sinh fulfill addition formulas similar to that one for sin and cos.

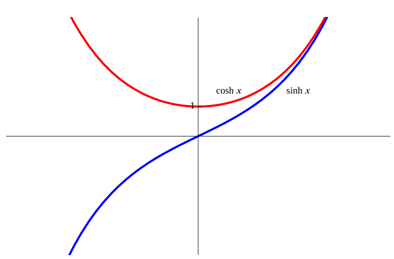
$\sinh(x+y) = \sinh x\cosh y + \sinh y\cosh x$, $\cosh(x+y) = \cosh x\cosh y - \sinh x\sinh y$, $\forall x,y\in\mathbb{R}$.

iv) $\cosh x, \sinh x \sim_{+\infty} \frac{e^x}{2}$, $\cosh x \sim_{-\infty} \frac{e^{-x}}{2}$, $\sinh x \sim_{-\infty} -\frac{e^{-x}}{2}$.
v) remarkable limits:

$$\lim_{x\to 0}\frac{\sinh x}{x} = 1,\quad \lim_{x\to 0}\frac{\cosh x - 1}{x^2} = \frac{1}{2}.$$

In particular: $\sinh x \sim_0 x$ and $\cosh x \sim_0 1 + \frac{x^2}{2}$.

Moreover $\sinh, \cosh \in \mathscr{C}(\mathbb{R})$.



Proof. i),ii), and iii) are easy checked by direct inspection. About iv) we have

$$\cosh x = \frac{e^x + e^{-x}}{2} = \begin{cases} \dfrac{e^x}{2}\left(1 + e^{-2x}\right) \sim \dfrac{e^x}{2}, \text{ because } 1 + e^{-2x} \longrightarrow 1, \; x \longrightarrow -\infty. \\[2ex] \dfrac{e^{-x}}{2}\left(1 + e^{2x}\right) \sim \dfrac{e^{-x}}{2}, \text{ because } 1 + e^{2x} \longrightarrow 1 \; x \longrightarrow -\infty. \end{cases}$$

Similarly for sinh. About v),

$$\frac{\sinh x}{x} = \frac{e^x - e^{-x}}{2x} = e^{-x}\frac{e^{2x} - 1}{2x} \longrightarrow 1, \; x \longrightarrow 0, \text{ by (5.5.3)}.$$

Inoltre

$$\frac{\cosh x - 1}{x^2} = \frac{\cosh x - 1}{x^2}\frac{\cosh x + 1}{\cosh x + 1} = \frac{(\cosh x)^2 - 1}{x^2}\frac{1}{\cosh x + 1} \overset{(5.7.1)}{=} \frac{(\sinh x)^2}{x^2}\frac{1}{\cosh x + 1} \longrightarrow \frac{1}{2}.$$

About continuity, finally, this depends on the continuity of exponentials. Indeed:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(e^x + \frac{1}{e^x}\right).$$

Accepting $e^x \in \mathscr{C}(\mathbb{R})$, then $\frac{1}{e^x} \in \mathscr{C}(\mathbb{R})$, hence also cosh. Similar argument for sinh. □

Example 5.7.2. *Compute*

$$\lim_{x\to 0} \frac{\log(2 - \cos x)}{\cosh x - 1}$$

Sol. — The limit is a form $\frac{0}{0}$: $\cos x \longrightarrow 1$, hence $2 - \cos x \longrightarrow 1$, so $\log(2 - \cos x) \longrightarrow \log 1 = 0$; the denominator goes to $\cosh 0 - 1 = 1 - 1 = 0$.

$$\frac{\log(2 - \cos x)}{\cosh x - 1} = \frac{\log(1 + (1 - \cos x)))}{\cosh x - 1} = \frac{\log(1 + (1 - \cos x))}{1 - \cos x}\frac{1 - \cos x}{x^2}\frac{x^2}{\cosh x - 1} \longrightarrow 1 \cdot \frac{1}{2} \cdot 2 = 1. \quad \square$$

## 5.8. Exercises

Exercise 5.8.1. Let $S := \{2^n \; : \; n \in \mathbb{Z}\}$. Then $\mathrm{Acc}(S) =$

□ $\emptyset$. □ $\{2^n \; : \; n \in \mathbb{Z}, \; n < 0\}$. □ $\{0\}$. □ $\{0, +\infty\}$.

Exercise 5.8.2. Let $S := \left\{(-1)^n \frac{1}{n} \; : \; n \in \mathbb{Z}\right\}$. Then $\mathrm{Acc}(S) =$

□ $\emptyset$. □ $[0, 1]$. □ $\{0\}$. □ $\{-1, 0, 1\}$.

Exercise 5.8.3. Let $S := [a, b]$ with $a < b$. Then $\mathrm{Acc}(S) =$

□ $[a, b]$. □ $]a, b[$. □ $\mathbb{R}$. □ none of the previous.

Exercise 5.8.4. Let $S := \left\{(1 + \cos(n\pi)) \frac{n+1}{n-1} \; : \; n \in \mathbb{N}, \; n \geqslant 2\right\}$. Then $\mathrm{Acc}(S) =$

□ $\{0, 1, 2\}$. □ $\{0, 1\}$. □ $\{0, 2\}$. □ $\{2\}$.

Exercise 5.8.5. Let $S := \{n + 2^n \; : \; n \in \mathbb{Z}\}$. Then $\mathrm{Acc}(S) =$

□ $\{-\infty, +\infty\}$. □ $\{0, +\infty\}$. □ $\{-\infty, 0, +\infty\}$. □ $\{-\infty, 0\}$.

Exercise 5.8.6. Classify indeterminate forms and compute:

1. $\displaystyle\lim_{x\to-\infty} \frac{x^3-5x+1}{x+2}$.
2. $\displaystyle\lim_{x\to+\infty} \frac{3x-2}{\sqrt{4x+1}+\sqrt{x+1}}$.
3. $\displaystyle\lim_{x\to2+} \frac{\sqrt{2}-\sqrt{x}}{x-2}$.
4. $\displaystyle\lim_{x\to-\infty} \left(2x+\sqrt{4x^2+x}\right)$.
5. $\displaystyle\lim_{x\to-1} \frac{x+1}{\sqrt{6x^2+3}+3x}$.
6. $\displaystyle\lim_{x\to+\infty} (\sqrt{9x^2+1}-3x)$.
7. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{2x^2+3}}{4x+2}$
8. $\displaystyle\lim_{x\to+\infty} \frac{\sqrt{x^4+1}+\sqrt[3]{x^5+x^2}}{\sqrt[4]{x^9-2x-1}-\sqrt[5]{x^7-x^5}}$.
9. $\displaystyle\lim_{x\to+\infty} \frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$.
10. $\displaystyle\lim_{x\to2+} \frac{|x-2|}{(x^2+1)(x-2)}$.
11. $\displaystyle\lim_{x\to0-} \frac{1}{1+x+\frac{|x|}{x}}$.
12. $\displaystyle\lim_{x\to1} \frac{x^2-1}{x^3-1}$.
13. $\displaystyle\lim_{x\to\pi} \frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\sin x}$.
14. $\displaystyle\lim_{x\to+\infty} \left(\sqrt{x+1}-\sqrt{x}\right)x$.

Exercise 5.8.7. Determine $a, b \in \mathbb{R}$ in such a way that $f \in \mathscr{C}(\mathbb{R})$ where

1. $f(x) := \begin{cases} \sin x, & x < \frac{\pi}{2}, \\ ax^2+b, & \frac{\pi}{2} \leqslant x < 2, \\ 3, & x \geqslant 2. \end{cases}$
2. $f(x) := \begin{cases} -2\sin x, & x < -\frac{\pi}{2}, \\ a\sin x+b, & -\frac{\pi}{2} \leqslant x < \frac{\pi}{2}, \\ \cos x, & x \geqslant \frac{\pi}{2}. \end{cases}$

Exercise 5.8.8. For each of the following functions i) find the domain $D$; ii) the subset $S \subset D$ where the function is continuous; iii) is it possible to extend $f$ to some point $x_0 \notin D$ in a continuous way?

1. $\dfrac{x^2-1}{x-1}$,
2. $\dfrac{\sin x}{x}$.
3. $\dfrac{\tan(2x)}{\tan x}$.
4. $\dfrac{\cos\left(\frac{\pi}{2}x\right)}{x^2-1}$.
5. $\dfrac{|x|}{x}$.
6. $\dfrac{1}{1+x+\frac{x}{|x|}}$.
7. $\sin \pi\dfrac{x}{|x|}$.
8. $x\log|x|$.
9. $\dfrac{\sqrt{x^4-3x^2+2}}{x}$.
10. $\dfrac{x\log x}{x-1}$.
11. $\log\left(\sqrt{1+x^2}-x\right)$.
12. $xe^{\frac{x}{x-1}}$.
13. $\sqrt{x+1}-2x+1$.
14. $\sin\left(\dfrac{e^x}{e^{2x}-e^x+1}\right)$.

Exercise 5.8.9. Discuss continuity of $f : [0,+\infty[\longrightarrow \mathbb{R}$, $f(x) := [x]+\sqrt{x-[x]}$.

Exercise 5.8.10. Say if the function

$$f(x) := e^{1/x}\sin x,\ x \neq 0,$$

can be extended by continuity at $x = 0$?

Exercise 5.8.11. Let $f : \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ defined as

$$f(x) := \begin{cases} a\cos x+\log(1-x), & x < 0, \\ \\ \sinh\frac{x}{x^2+1} - a\cos\frac{1}{\sqrt{x}}, & x > 0. \end{cases}$$

Are there values of $a \in \mathbb{R}$ such that $f$ can be extended by continuity at $x = 0$?

Exercise 5.8.12 ($\star$). Determine all possible $a, b \in \mathbb{R}$ such that

$$\lim_{x\to+\infty}\left(\sqrt{x^4+x^2+1}-(ax^2+bx)\right)\in\mathbb{R}.$$

Exercise 5.8.13. Order in increasing way (with respect to $\ll_{+\infty}$) the following quantities:

$$2^{2^{2x}},\quad x^{2^x},\quad x\sqrt{x},\quad 2^{\log x+\log(\log x)},\quad x^{1+\frac{1}{\sqrt{\log x}}}.$$

Exercise 5.8.14. Compute

1. $\displaystyle\lim_{x\to+\infty}\frac{e^x+e^{2x}}{2^x+e^{x^2}}$.
2. $\displaystyle\lim_{x\to0+}x^x$.
3. $\displaystyle\lim_{x\to+\infty}x^{1/x}$.
4. $\displaystyle\lim_{x\to+\infty}\frac{\log x+x^2}{\sqrt{x}+x\log x+x^2}$.
5. $\displaystyle\lim_{x\to+\infty}\frac{x\log x}{(\log x-1)^2}$.
6. $\displaystyle\lim_{x\to+\infty}\frac{\log(\log(\log x))}{(\log(\log x))^{\alpha}}$, $(\alpha>0)$.
7. $\displaystyle\lim_{x\to0+}\frac{\log x-1}{\sqrt{x}\log^4 x}$
8. $\displaystyle\lim_{x\to+\infty}\frac{x^{\log(x^2)}+\cos x}{4^x+\pi^x}$.
9. $\displaystyle\lim_{x\to+\infty}\frac{(\log x)^x+\sin(x^x)}{6^x+x^2\log x}$.
10. $\displaystyle\lim_{x\to0+}\frac{x^{x^x}}{x}$.
11. $\displaystyle\lim_{x\to0+}\frac{1}{x}e^{-x^2\log x}$
12. $\displaystyle\lim_{x\to+\infty}\left(\frac{\log x}{x}\right)^{1/x}$.
13. $\displaystyle\lim_{x\to+\infty}\frac{xe^x-e^{2\sqrt{x^2+1}}}{e^{2x}+x^4}$

Exercise 5.8.15. Reducing to fundamental limits, compute the following limits:

1. $\displaystyle\lim_{x\to0}\frac{1-\cos x}{1-\cos\frac{x}{2}}$.
2. $\displaystyle\lim_{x\to0}\frac{1-(\cos x)^3}{(\sin x)^2}$.
3. $\displaystyle\lim_{x\to1}\frac{\sin(\pi x^2)}{x-1}$.
4. $\displaystyle\lim_{x\to0}\frac{\sqrt{x^2-x+1}-1}{\sin x}$
5. $\displaystyle\lim_{x\to0}\frac{\log(1+x)}{\tan x}$.
6. $\displaystyle\lim_{x\to0}\frac{x^3}{\tan x-\sin x}$.
7. $\displaystyle\lim_{x\to0}\frac{e^{\sin(3x)}-1}{\sinh(2x)}$.
8. $\displaystyle\lim_{x\to0}\frac{\cos x-\cos(2x)}{1-\cos x}$.
9. $\displaystyle\lim_{x\to+\infty}\left(\frac{x+2}{x+1}\right)^x$.
10. $\displaystyle\lim_{x\to1}x^{\frac{2}{x-1}}$.
11. $\displaystyle\lim_{x\to+\infty}\left(\frac{1}{\log x}\right)^{\frac{\log x}{x}}$.
12. $\displaystyle\lim_{x\to0+}\left(\frac{1}{x}\right)^{\tan x}$.
13. $\displaystyle\lim_{x\to+\infty}\frac{x\left(x^{\frac{1}{x}}-1\right)}{\log x}$.
14. $\displaystyle\lim_{x\to0+}(\log x)(\log(1-x))$.
15. $\displaystyle\lim_{x\to1}x^{\frac{x}{x-1}}$.
16. $\displaystyle\lim_{x\to0+}e^{\frac{1}{x}}\tan x$.
17. $\displaystyle\lim_{x\to\frac{\pi}{2}}(1+\cos x)^{\tan x}$.
18. $\displaystyle\lim_{x\to+\infty}\left(1+\frac{x+1}{x^2}\right)^x$.

19. $\displaystyle\lim_{x\to+\infty}\left(1+\frac{\arctan x}{x}\right)^x$.  20. $\displaystyle\lim_{x\to 0}\frac{\log\cos x}{\sqrt[4]{1+x^2}-1}$.  21. $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{\sin(4x)}$.

22. $\displaystyle\lim_{x\to 0}\frac{e^{\sin(3x)}-1}{x}$.  23. $\displaystyle\lim_{x\to 0}\frac{\log(2-\cos(2x))}{x^2}$.  24. $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+\sin(3x)}-1}{\log(1+\tan(2x))}$.

25. $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{1-\cos x}$.  26. $\displaystyle\lim_{x\to 0}\frac{\sqrt[4]{\cos x}-1}{\log(1+x\sin x-x^3)}$  27. $\displaystyle\lim_{x\to 0}\frac{\log(1+2x-3x^2+4x^3)}{\log(1-x+2x^2-3x^3)}$.

28. $\displaystyle\lim_{x\to+\infty}\left(1+\frac{x+1}{x^2}\right)^x$.  29. $\displaystyle\lim_{x\to+\infty}\left(\frac{x^2+x+1}{x^2+1}\right)^x$.  30. $\displaystyle\lim_{x\to 0}\frac{(e^{2x^2+x^3}-1)\sin x}{\sqrt[3]{1+x^3}-1}$.

31.($\star$) $\displaystyle\lim_{x\to 1+}\frac{x^x-1}{(\log x-x+1)^2}$.  32.($\star$) $\displaystyle\lim_{x\to 0-}(1-e^x)^{\sin x}$.

CHAPTER 6

# Continuity

## 6.1. Class of continuous functions

In the previous chapter, we introduced the concept of *continuous function* at some point and on some set (that is, at every point of the set). In this chapter we explore better this concept. We recall that

$$f : D \subset \mathbb{R} \longrightarrow \mathbb{R},\ \xi \in D \cap \mathrm{Acc}(D),\ f \text{ continuous at } \xi,\ \iff \lim_{x\to\xi} f(x) = f(\xi).$$

If $f$ is continuous at every point of a set $S$ we write $f \in \mathscr{C}(S)$, the class of continuous functions on the set $S$.

EXAMPLE 6.1.1. $x^n \in \mathscr{C}(\mathbb{R})$, $n \in \mathbb{N}$. Indeed, clearly $\lim_{x\to x_0} x^n = x_0^n$. □

EXAMPLE 6.1.2. $\sin, \cos \in \mathscr{C}(\mathbb{R})$.

PROOF. We start with the continuity of sin at $x_0 = 0$. Recall that, by (5.5.5),

$$0 \leqslant \sin x \leqslant x,\ \forall x \in \left[0, \frac{\pi}{2}\right[.$$

Letting $x \longrightarrow 0+$ and applying the two policemen theorem, we obtain $\lim_{x\to 0+} \sin x = 0 = \sin 0$. Finally,

$$\lim_{x\to 0-} \sin x \overset{y=-x}{=} \lim_{y\to 0+} \sin(-y) = -\lim_{y\to 0+} \sin y = 0.$$

Next, we show continuity of cos at $x = 0$. We have

$$1 - \cos x = (1 - \cos x)\frac{1 + \cos x}{1 + \cos x} = \frac{1 - (\cos x)^2}{1 + \cos x} = \frac{(\sin x)^2}{1 + \cos x}.$$

Now, since $x \longrightarrow 0$ we may assume, for instance, $|x| \leqslant \frac{\pi}{4}$. Therefore $\cos x \geqslant \frac{\sqrt{2}}{2}$ thus

$$0 \leqslant 1 - \cos x \leqslant \frac{1}{1 + \frac{\sqrt{2}}{2}}(\sin x)^2,$$

and letting $x \longrightarrow 0$ the conclusion now follows by the police theorem theorem.

Finally, we prove continuity of sin (and similarly for cos) at $\xi$. Notice that

$$\lim_{x\to\xi} \sin x \overset{h=x-\xi}{=} \lim_{h\to 0} \sin(\xi + h) = \lim_{h\to 0} (\sin\xi\cos h + \sin h\cos\xi) = \sin\xi. \quad □$$

$$\lim_{x\to\xi} \cos x \overset{h=x-\xi}{=} \lim_{h\to 0} \cos(\xi + h) = \lim_{h\to 0} (\cos\xi\cos h - \sin h\sin\xi) = \cos\xi. \quad □$$

An immediate consequence of the rules of calculus for limits is

**Proposition 6.1.3**

Let $f, g$ be continuous at $\xi$. Then

i) $f \pm g.\ f \cdot g$ are continuous at $\xi$;

ii) $\frac{f}{g}$ is continuous at $\xi$ provided $g(\xi) \neq 0$.

Proof may be easily done as exercise.

Example 6.1.4. *Polynomials are continuous functions on* $\mathbb{R}$. Indeed, a polynomial is a sum of monomials of type $c_k x^k$. Since $c_k, x^n \in \mathscr{C}(\mathbb{R})$ then $c_k x^k \in \mathscr{C}(\mathbb{R})$, thus the polynomial itself is in $\mathscr{C}(\mathbb{R})$. □

Example 6.1.5. $x^n \in \mathscr{C}(\mathbb{R}\backslash\{0\})$, $\forall n \in \mathbb{Z}$, $n < 0$.

Sol. — Indeed, we may write $n = -m$ with $m \in \mathbb{N}$, $m > 0$ and

$$x^n = x^{-m} = \frac{1}{x^m}.$$

Now: $1, x^m \in \mathscr{C}(\mathbb{R})$ then, by ii), $\frac{1}{x^m} \in \mathscr{C}(\mathbb{R}\backslash\{0\})$. □

Example 6.1.6.

$$\tan \in \mathscr{C}\left(\mathbb{R}\backslash\left\{\frac{\pi}{2} + k\pi\ :\ k \in \mathbb{Z}\right\}\right).$$

Sol. — We have $\tan := \frac{\sin}{\cos}$, so tan is continuous where $\cos \neq 0$, that is on $\mathbb{R}\backslash\{\frac{\pi}{2} + k\pi\ :\ k \in \mathbb{Z}\}$. □

There is another important operation on functions: the *composition*.

**Proposition 6.1.7**

Let $h : D \to \mathbb{R}$, $g : E \to \mathbb{R}$ be two functions, with $h(D) \subset E$, let $f := g \circ h : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ their composition, and let us consider a point $\xi \in D \cap \mathrm{Acc}(D)$ such that $y_0 = h(\xi) \in \mathrm{Acc}(E)$ [1]. Suppose that

i) $h$ is continuous at $\xi$, and

ii) $g$ is continuous at $y_0 = h(\xi)$.

Then, the composition $f = g \circ h$ is continuous at $\xi$.

This proposition is a straightforward consequence of the following particular case of the result on change of variable in limits:

[1] We already know that $y_0 = h(\xi) \in E$, so $y_0 \in Acc(E) \cap E$.

**Proposition 6.1.8: Change of variable with $g$ continuous**

Assume that $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\xi \in \mathrm{Acc}D$. Suppose that there exists a neighbourhood $I_\xi$ of $\xi$, and functions $h : D \longrightarrow \mathbb{R}$, $g : E \to \mathbb{R}$ such that

i) $h(I_\xi \setminus \{\xi\}) \subseteq E$, and $f(x) = g(h(x))$, $\forall x \in I_\xi \setminus \{\xi\}$;

ii) $\lim_{x \to \xi} h(x) = y_0$;

iii) $y_0 \in \mathrm{Acc}(h(D)) \cap h(D)$ and $g$ is continuous at $y_0$

Then

$$\Longrightarrow \quad \exists \lim_{x \to \xi} f(x) = \lim_{x \to \xi} g(h(x)) = g(y_0).$$

Proof. Let $(x_n)$ be a sequence converging to $\xi$. In particular, there exists $N \in \mathbb{R}$ such that $x_n \in I_\xi$ for every $n \geq N$. Therefore, by ii) one gets $y_n := h(x_n) \longrightarrow y_0$ (as $n$ tends to $\infty$), so that, by the continuity of $g$ at $y_0$, $g(y_n) \longrightarrow g(y_0)$. Hence, by i) $f(x_n) = g(h(x_n)) = g(y_n) \longrightarrow g(y_0)$. In view of the fact that $(x_n)$ is arbitrary, this concludes the proof. □

Remark 6.1.9. The continuity of $g$ at $y_0$ allows us to omit the assumption $h(x) \neq y_0$ for all $x \in I_\xi \setminus \{\xi\}$, which instead is crucial in the general version of the result on change of variable (see Proposition 5.4).

Proof of Proposition 6.1. It is a consequence of the change of variable established in Proposition 6.1:

$$\lim_{x \to \xi} f(x) = \lim_{x \to \xi} (g \circ h)(x) = \lim_{x \to \xi} g(h(x)) \overset{y = f(x)}{=} \lim_{y \to h(\xi)} g(y) = g(h(\xi)) = (g \circ h)(\xi) = f(\xi). \quad \square$$

Example 6.1.10. *Find the domain D and the continuity set of*

$$f(x) = \tan \frac{1}{x}.$$

Sol. — Clearly $D = \{x \in \mathbb{R} \ : \ x \neq 0, \ \frac{1}{x} \neq \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z}\}$. Thus, $x \neq 0$ and $x \neq \frac{1}{\frac{\pi}{2} + k\pi}$, $k \in \mathbb{Z}$. Since $h$ is the composition of the two continuous functions $h(x) = \frac{1}{x}$ and $g(y) = \tan y$, we conclude $f$ is continuous at every point of its domain $D$. □

Remark 6.1.11 (Warning!). The composition rule says,

$$f \text{ cont. at } \xi, \ g \text{ cont. at } f(\xi) \quad \Longrightarrow \quad g \circ f \text{ cont. at } \xi.$$

A frequent error is to think that $\Longrightarrow$ is actually an $\Longleftrightarrow$. That is, one may think that **to have** $g \circ f$ **continuous at some point** $\xi$ **then, necessarily,** $f$ **must be continuous at** $\xi$ **and** $g$ **must be continuous at** $f(\xi)$. Thus, in particular, if either $f$ is not continuous at $\xi$ or $g$ is not continuous at $f(\xi)$, then $g \circ f$ cannot be continuous at $\xi$. This is completely false!

For example, take $f$ any non continuous function at $\xi$ and let $g(y) \equiv C$ (constant). Then clearly $g \circ f \equiv C$ is continuous at $\xi$ despite $f$ is not continuous at $\xi$. □

## 6.2. Monotonic functions

Here we start to discuss the properties of continuous functions on intervals. The following result provides the basis for proving the continuity of many elementary functions such as power (rational and real exponents), exponential, and logarithm.

**Theorem 6.2.1**

Let $I, J$ be intervals and let us consider a function $f : I \longrightarrow J$ If $f$ is surjective (that is, $J = f(I)$) and monotonic, then $f \in \mathscr{C}(I)$.

Proof. We consider the case $f \nearrow$ the other is similar. Suppose that $f$ is not continuous at some $\xi \in I$. Since $f$ is monotonic, there exists

$$f(\xi-) := \lim_{x\to\xi-} f(x) = \sup\{f(x) \ : \ x < \xi\}, \quad f(\xi+) := \lim_{x\to\xi+} f(x) = \inf\{f(x) \ : \ x > \xi\}.$$

In general $f(\xi-) \leqslant f(\xi) \leqslant f(\xi+)$. Since we are assuming $f$ not continuous at $\xi$, necessarily one of these inequalities must be strict.

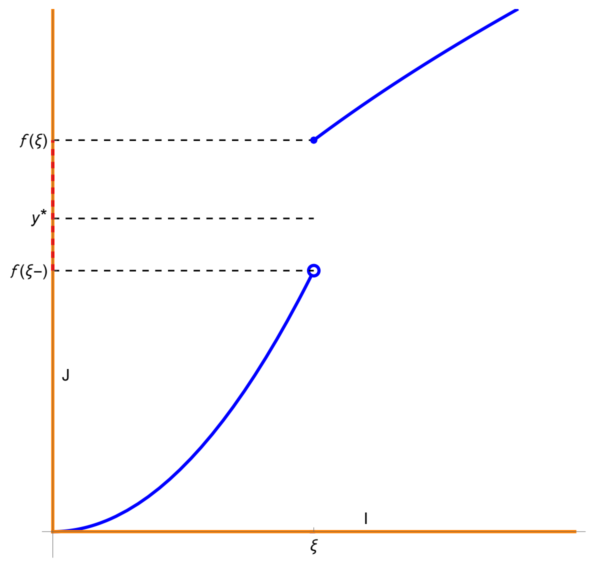
Assume, for instance, that

$$f(\xi-) < f(\xi).$$

As we can see in the figure, there should be a "jump" at point $\xi$, and this should conflicts with surjectivity of $f$. Indeed: pick $y^* \in ]f(\xi-), f(\xi)[\subset J$. We claim that there is no $x^*$ such that $y^* = f(x^*)$. This would be a contradiction because $y^* \in J$ and $f(I) = J$ by our assumptions. Indeed: if such $x^*$ exists, we may have

- $x^* < \xi$, but then, being $f \nearrow$, $f(x^*) \leqslant f(\xi-) < y^*$;
- $x^* = \xi$, but then $f(x^*) = f(\xi) > y^*$;
- $x^* > \xi$, but again $f(x^*) \geqslant f(\xi+) \geqslant f(\xi) > y^*$.

This completes the proof. □




Since power, exponential, and logarithm are monotonic, surjective, functions, by the previous theorem, we obtain their continuity:

**Corollary 6.2.2**

Power (any exponent), exponential (any base), and logarithms (any base) are continuous functions in their natural domains.

A strictly increasing function $f : I \longrightarrow J = f(I)$ is, at once, both injective (since for $x < y$ we have $f(x) < f(y)$) and surjective (because we setted $J = f(I)$). Thus, it is invertible and $f^{-1} : J \longrightarrow I$ is well defined and strictly monotonic (as $f$). By the previous theorem, we obtain automatically that $f^{-1}$ is continuous:

**Corollary 6.2.3**

Let $f : I \longrightarrow J$ strictly monotone and surjective between $I$ and $J$ intervals. Then $f, f^{-1}$ are both continuous on their domains.

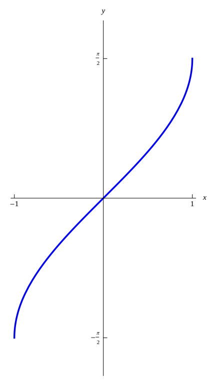
EXAMPLE 6.2.4 (arcsin, arccos). Globally, sin and cos are not invertible functions. However, when we restrict them to specific intervals, they become invertible functions. Consider

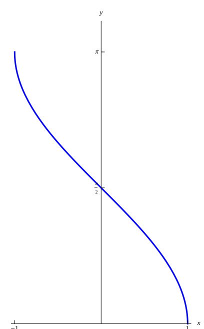
$$f := \sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1].$$

Then sin $\nearrow$ strictly and $\sin^{-1} =:$ arcsin is well defined and continuous:

$$\arcsin := \sin^{-1} : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad \arcsin \in \mathscr{C}([-1, 1]).$$

Similarly is defined arccos, inverting $\cos : [0, \pi] \longrightarrow [-1, 1]$. Below graphs of arcsin and arccos.





EXAMPLE 6.2.5 (arctan). As sin and cos, tan is not invertible. However, considering

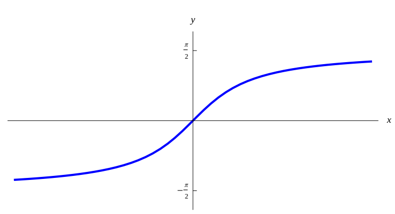
$$\tan : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[ \longrightarrow ]-\infty, +\infty[,$$

tan $\nearrow$ strictly and

$$\arctan := \tan^{-1} : ]-\infty, +\infty[ \longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[, \ \arctan \in \mathscr{C}(\mathbb{R}).$$

Clearly $\arctan 0 = \tan^{-1} 0 = 0$, and

$$\lim_{x \to -\infty} \arctan x \overset{y=\arctan x,\ x=\tan y}{=} \lim_{y \to -\frac{\pi}{2}} y = -\frac{\pi}{2}, \quad \lim_{x \to +\infty} \arctan x = \frac{\pi}{2}.$$



Example 6.2.6 ($\sinh^{-1}$, $\cosh^{-1}$). The function $\sinh : \mathbb{R} \longrightarrow \mathbb{R}$ is strictly increasing, how can be easily checked. Its inverse $\sinh^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. We may notice that

$$\sinh^{-1} y = x, \iff \sinh x = y, \iff \frac{e^x - e^{-x}}{2} = y, \iff e^x - \frac{1}{e^x} = 2y, \iff e^{2x} - 2ye^x - 1 = 0.$$

This gives

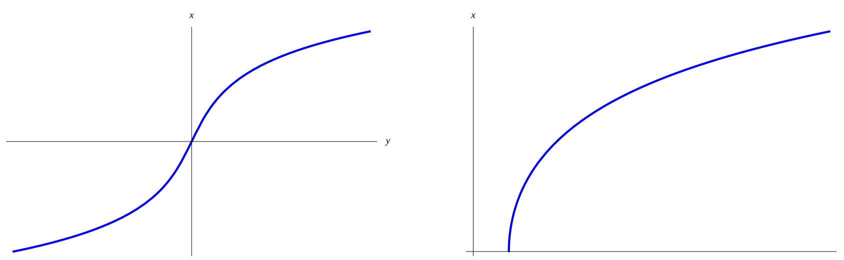
$$e^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1}.$$

Now, $y - \sqrt{y^2 + 1}$ is always negative (evident for $y < 0$, easy for $y \geqslant 0$) so the unique possibility is

$$\sinh^{-1} y = \log\left(y + \sqrt{y^2 + 1}\right). \tag{6.2.1}$$

The hyperbolic cosine is not globally invertible on $\mathbb{R}$, but $\cosh : [0, +\infty[ \longrightarrow [1, +\infty[$ is strictly increasing and surjective (easy): by the continuous inverse mapping thm it is well defined and continuous the function $\cosh^{-1} : [1, +\infty[ \longrightarrow [0, +\infty[$. In this case

$$\cosh^{-1} y = \log\left(y + \sqrt{y^2 - 1}\right). \tag{6.2.2}$$




Above graphs of $\sinh^{-1}$ (left) and $\cosh^{-1}$ (right). □

## 6.3. Zeroes theorem

Several problems lead to the solution of a certain equation

$$f(x) = 0.$$

It is therefore important to have methods to determine the solutions of such equations. We cannot expect, in general, to have a method to solve explicitly any equation. Actually, even the existence of a solution is almost never a straightforward fact.

Solutions of $f(x) = 0$ are *zeroes* of $f$, so the problem of finding a solution to certain equation may be revisited as the problem of finding where $f$ can take value $= 0$. Continuity plays an essential role in this problem as we will see now. An important feature of the following result is that it provides an algorithm (known as *bisection method*) for the search of zeroes:

**Theorem 6.3.1: Bolzano**

Let $f \in \mathscr{C}(I)$, $I \subset \mathbb{R}$ interval, be such that $f(a) < 0$ and $f(b) > 0$ for some $a, b \in I$: then
$$\exists\, c \in I \ :\ f(c) = 0.$$
It turns out that
$$c = \lim_{n \to +\infty} c_n,$$
where $(c_n)$ is defined by:
$$c_n = \frac{a_n + b_n}{2},\ a_0 = a,\ b_0 = b,\ a_{n+1} = \begin{cases} a_n & \text{if } f(a_n) < 0, \\ \frac{a_n+b_n}{2} & \text{if } f(a_n) \geqslant 0, \end{cases} \quad b_{n+1} = \begin{cases} b_n & \text{if } f(b_n) > 0, \\ \frac{a_n+b_n}{2} & \text{if } f(b_n) \leqslant 0. \end{cases}$$

Proof. Suppose, for instance, $a < b$. Consider the mean point between $a$ and $b$, that is $\frac{a+b}{2}$. We have the following alternatives:

i) $f\left(\frac{a+b}{2}\right) = 0$, we are lucky and the proof is finished.

ii) $f\left(\frac{a+b}{2}\right) > 0$: we restrict our search to the interval $[a, \frac{a+b}{2}] =: [a_1, b_1]$.

iii) $f\left(\frac{a+b}{2}\right) < 0$: we restrict our search to the interval $[\frac{a+b}{2}, b] =: [a_1, b_1]$.

In cases ii) iii), we have
$$a \leqslant a_1 \leqslant b_1 \leqslant b,\ \ b_1 - a_1 = \frac{b-a}{2},\ f(a_1) < 0 < f(b_1).$$
Let's repeat the argument on the interval $[a_1, b_1]$. Consider again the mean point: either we find a zero, or there is a new interval $[a_2, b_2]$ such that
$$a \leqslant a_1 \leqslant a_2 \leqslant b_2 \leqslant b_1 \leqslant b,\ \ b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b-a}{2^2},\ f(a_2) < 0 < f(b_2).$$
Iterating the procedure we have the following alternative: either we stop after a certain number of steps because we found a zero, or we will never end, constructing and infinite family of intervals $[a_n, b_n]$ such that
$$a \leqslant \ldots \leqslant a_{n-1} \leqslant a_n \leqslant b_n \leqslant b_{n-1} \leqslant \ldots \leqslant b,\ \ b_n - a_n = \frac{b-a}{2^n},\ f(a_n) < 0 < f(b_n).$$
Being $a_n \nearrow$ and $b_n \searrow$ they both have a limit and since
$$0 \leqslant b_n - a_n \leqslant \frac{b-a}{2^n} \longrightarrow 0,$$
we get
$$a_n, b_n \longrightarrow c \in I.$$
Moreover, by continuity and permanence of sign
$$f(c) = \lim_n f(a_n) \leqslant 0,\ \ f(c) = \lim_n f(b_n) \geqslant 0,$$
and by this, necessarily, $f(c) = 0$. $\square$

Remark 6.3.2. The procedure described in the proof can easily be translated into a code for the search of a solution to
$$f(x) = 0.$$

Since at each step the search is restricted to an interval of length $\frac{b-a}{2^n}$ and this decreases rapidly to 0, if one takes $a_n$ or $b_n$ as approximate zero, the true zero $c$ will be at maximum distance $\frac{b-a}{2^n}$. In particular: if we fix an error $\varepsilon > 0$, by taking $N$ in such a way that

$$\frac{b-a}{2^N} \leqslant \varepsilon, \iff N = \left[\log_2 \frac{b-a}{\varepsilon}\right] + 1,$$

by choosing $a_N$ or $b_N$ we have an approximation of $c$ with an error $\leqslant \varepsilon$. Notice that this $N$ depends on $a, b$ and $\varepsilon$ but not by $f$.

For example: consider the equation

$$x^3 + 3x - 1 = 0.$$

Let $f$ the lhs, $f(0) = -1$, $f(1) = 3$, thus, according to zeroes thm, there is a zero somewhere in $[0, 1]$. Suppose we wish to determine it with an error $\leqslant \frac{1}{10^5}$. According to the previous discussion, we have to iterate the algorithm

$$N = \left[\log_2 \frac{1}{1/10^5}\right] + 1 = 5\log_2 10 + 1 = 17$$

times. Applying the algorithm, we find an approximate value for the solution $c = 0,322185$. □

A straightforward consequence of Zeores theorem is

**Corollary 6.3.3: intermediate values**

Let $f \in \mathscr{C}(I)$, $I \subset \mathbb{R}$ interval. Let $\alpha < \beta$ be such that $\alpha, \beta \in f(I)$ (that $\alpha, \beta$ are values assumed by $f$). Then

$$\forall \gamma \in ]\alpha, \beta[ \implies \gamma \in f(I).$$

Proof. Since $\alpha, \beta \in f(I)$, there exists $x_1, x_2 \in I$ such that $\alpha = f(x_1)$ and $\beta = f(x_2)$. If one defines the function $g : I \longrightarrow \mathbb{R}$ by setting $g(x) = f(x) - \gamma$ $\forall x \in I$ one gets

$$g(x_1) = \alpha - \gamma < 0, \qquad g(x_2) = \beta - \gamma > 0.$$

Therefore by Bolzano's Theorem (Th. 6.3) we deduce the existence of a point $\bar{x}$ such that

$$0 = g(\bar{x}) = f(\bar{x}) - \gamma,$$

so that $f(\bar{x}) = \gamma$, namely $\gamma \in f(I)$. □

### 6.4. Existence of minima and maxima: Weierstrass' Theorem

One of the most relevant applied and theoretical problems is the search for the minimum/maximum of a numerical function. Let us formalize these concepts:

**Definition 6.4.1**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$. We say that $x_{max} \in D$ is **a global maximum point for** $f$ **on** $D$ if

$$f(x_{max}) \geqslant f(x), \ \forall x \in D.$$

We call $f(x_{max}) = \max_D f$ the **maximum (value) of** $f$ **on** $D$.
Similarly, we say that $x_{min} \in D$ is **a global minimum point for** $f$ **on** $D$ if

$$f(x_{min}) \leqslant f(x), \ \forall x \in D.$$

We call $f(x_{min}) = \min_D f$ the **minimum (value) of** $f$ **on** $D$.

Remark 6.4.2. Global maximum points and global minimum points are also called **absolute maximum points** and **absolute minimum points**, respectively.

Existence and search method for min/max points is an extremely relevant problem. For numerical functions, the existence follows under relatively large assumptions, verified in many concrete situations. This is the content of Weierstrass' theorem.

**Theorem 6.4.3: Weierstrass**

Any continuous function on a closed and bounded interval has a global min/max on it.

Proof. The proof is subtle and based on Bolzano–Weiestrass Thm 4.3.

**First step:** $f([a, b])$ **is bounded.** Suppose, on the contrary, that $f([a, b])$ is unbounded, for instance from above,

$$\nexists\, K > 0, \ : \ f(x) \leqslant K, \ \forall x \in [a, b], \ \Longleftrightarrow \ \forall K > 0, \ \exists x \in [a, b], \ : \ f(x) \geqslant K.$$

Take $K = n \in \mathbb{N}$ and call $x_n \in [a, b]$ such that $f(x_n) \geqslant n$. Then $(x_n) \subset [a, b]$ and by the Bolzano–Weierstrass Thm. 4.3 there exists $(x_{n_k}) \subset (x_n)$ such that $x_{n_k} \longrightarrow \xi$. Notice that $\xi \in [a, b]$. [2]So, on the one hand,

$$f(x_{n_k}) \longrightarrow +\infty$$

while, on the other hand

$$f(x_{n_k}) \longrightarrow f(\xi),$$

in that $f$ is continuous. It follows that $f(\xi) = +\infty$, which is a nonsense, being $f(\xi) \in \mathbb{R}$.

**Second step: existence of** min/max**.** Let's consider the case of max. Call

$$M := \sup \{f(x) \ : \ x \in [a, b]\} .$$

Being $f$ bounded, $M < +\infty$. Let us prove that there exists $x_{max} \in [a, b]$ such that $f(x_{max}) = M$: if this is true, we are done! Since $M$ is the sup,

$$\forall n \in \mathbb{N}, \ \exists x_n \in [a, b], \ : \ M - \frac{1}{n} \leqslant f(x_n) \leqslant M.$$

[2]If, by contradiction, it were $\xi \notin [a, b]$, one would have $d := d(\xi, [a, b]) = \min_{x \in [a,b]} |x - \xi| > 0$, so that $|x_{n_k} - \xi| \geq d$ for all $k \in \mathbb{N}$. Therefore $x_{n_k}$ would be not allowed to converge to $\xi$.

Again: $(x_n) \subset [a,b]$, by Bolzano–Weierstrass there exists $(x_{n_k}) \subset (x_n)$ such that $x_{n_k} \longrightarrow x_{max} \in [a,b]$. Then $f(x_{n_k}) \longrightarrow f(x_{max})$ so

$$M \leqslant f(x_{max}) \leqslant M, \implies f(x_{max}) = M. \quad \square$$

It is easy to see that both assumptions ($f$ continuous and $I$ closed and bounded intervals) are essential, if we remove one of these, we can easily find counterexamples. Weierstrass' theorem does not provide a method for the search of min/max points. This method will be provided as the consequences of Differential Calculus, the protagonist of the next chapter.

## 6.5. Exercises

EXERCISE 6.5.1. Give an example of functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ such that

i) $f$ is continuous on $\mathbb{R}$, $g$ is not continuous on $\mathbb{R}$ but $g \circ f$ is continuous on $\mathbb{R}$.
ii) both $f$ and $g$ are not continuous on $\mathbb{R}$ but $g \circ f$ is continuous on $\mathbb{R}$.

EXERCISE 6.5.2. Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) := \begin{cases} 0, & x \in \mathbb{Q}, \\ \\ 1, & x \notin \mathbb{Q}. \end{cases}$$

Discuss the continuity of $f$ at a generic $x_0 \in \mathbb{R}$. (recall density of $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ in $\mathbb{R}$...)

EXERCISE 6.5.3. Consider the equation $3x^3 - 8x^2 + x + 3 = 0$. Show that there are three solutions, respectively, in $]-\infty, 0[$, $]0,1[$ and $]1,+\infty[$. Determine that one in $]0,1[$ with an approximation of $\frac{1}{10}$.

EXERCISE 6.5.4. Let $p$ be an $n-$th degree polynomial with $n$ odd. Show that the equation $p(x) = 0$ has at least one solution.

EXERCISE 6.5.5. Let $f : I \longrightarrow I$ be a continuous function on $I = [a,b]$. Show that there exists $c$ such that $f(c) = c$. Is it true that conclusion holds if $I = ]a,b[$?

EXERCISE 6.5.6 ($\star$). Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous on $\mathbb{R}$. Suppose that $\lim_{x \to \pm\infty} f(x) = 0$. Show that, if $f$ is not identically 0, then $f$ has at least one among $\max f$ or $\min f$ different from 0. Is it always true that both are $\neq 0$? Prove or exhibit a counterexample.

EXERCISE 6.5.7 ($\star$). Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous on $\mathbb{R}$. Suppose that $\lim_{x \to \pm\infty} f(x) = +\infty$. Show that $f$ has global min over $\mathbb{R}$.

EXERCISE 6.5.8 ($\star$). Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous and injective. Show that if $f(a) < f(b)$ then $f(a) < f(x) < f(b)$ for all $x \in ]a,b[$. Can you say that $f$ is strictly increasing?

CHAPTER 7

# Differential Calculus

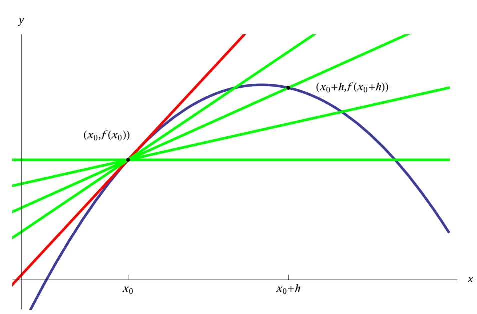
## 7.1. What is Differential Calculus?

It is not an exaggeration to say that *Differential Calculus* is one of the major creations of human thought. Its considerable number of applications reveals its importance: from the entire Mathematics to Physics and all applied science such as Engineering, Natural Sciences, Economy. The idea, common to many problems, that a certain behaviour (natural or human) follows an *optimization principle* is deep and beautiful. Consequently, to be able to solve optimization problems has paramount importance. Calculus offers an extremely powerful tool to solve this type of problems when optimizing variables belonging to a continuum.

The origin of *Calculus* (as Newton used to call it) has its roots in a geometrical problem: *given a plane curve, determine its tangent at a given point*. There are many ways we can formalize the concept of *curve*. A possible (but not the most general way!) is through the *graph* of a function $y = f(x)$. In this case, the curve is the set of points $\{(x, f(x)) \ : \ x \in D\}$. Here we will consider this last case. Let $(x_0, f(x_0))$ be a point on the curve/graph. There are, of course infinitely many straight lines passing through this point. Excluding the particular case when the line is vertical, the remaining cases can be described through a Cartesian equation

$$y = m(x - x_0) + f(x_0), \ x \in \mathbb{R},$$

the number $m$ being the *angular coefficient*. Therefore, we may rephrase our problem as that of searching for an $m$ in such a way the straight line described by the previous equation be *tangent* to the graph of $f$. The question is: *how can we determine such an $m$?*




Since for the moment it is not clear what does it mean *tangent*, we do some heuristic consideration to guess the answer. Pick a second point on the graph, say $(x_0 + h, f(x_0 + h))$ with $h \neq 0$. In this way, $(x_0 + h, f(x_0 + h)) \neq (x_0, f(x_0))$, therefore there is a unique straight line passing through these points.

This line has an angular coefficient

$$m_h = \frac{\text{ordinate variation}}{\text{abscissa variation}} = \frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0} = \frac{f(x_0+h)-f(x_0)}{h}.$$

We may now notice that as $h \longrightarrow 0$ the straight lines rotates as hinged on $(x_0, f(x_0))$ and tends to assume a "limit" position that should correspond to the tangent to $f$ at point $(x_0, f(x_0))$ (we simply say at $x_0$). The corresponding angular coefficient should be then

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}, \tag{7.1.1}$$

provided, of course, this limit exists. In this case, the limit is called *derivative of $f$ at point $x_0$* and it is usually denoted by one of the following notations

$$f'(x_0), \quad \text{(Newton)}, \quad \frac{df}{dx}(x_0), \quad \text{(Leibniz)}.$$

Derivative seems to be tightly connected to the problem of search for min/max of $f$ and not only. Indeed, the first rough remark is that at the min/max point for $f$, the tangent seems to be horizontal, that is, $f' = 0$. This is not 100% true, but it shows that we are on the right track.

## 7.2. Definition and first properties

Before we define properly limit (7.1.1) we need to say something on the points at which that limit makes sense. First: since we have to compute $f(x_0)$, we need $x_0 \in D$. Second: we have to compute $f(x_0+h)$ for $h \longrightarrow 0$, that is we need to compute $f$ at points $x_0+h$, at least for $h$ small. This means that an interval $[x_0-\varepsilon, x_0+\varepsilon]$ should be contained in the domain of $f$. This deserve a special

**Definition 7.2.1**

Given $D \subset \mathbb{R}$ we say that $x_0$ in an **interior** point for $D$ if

$$\exists \varepsilon > 0, \,:\, I_{x_0} = [x_0-\varepsilon, x_0+\varepsilon] \subset D.$$

The set of all interior points of $D$ is denoted by $\mathrm{Int}(D)$.

Thus, for example

$$\mathrm{Int}([a,b]) = \mathrm{Int}(]a,b[) = \mathrm{Int}([a,b[) = \mathrm{Int}(]a,b]) = ]a,b[,$$

but $a, b \notin \mathrm{Int}([a,b])$.

**Definition 7.2.2: derivative**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $x_0 \in \mathrm{Int}(D)$. We say that $f$ **is differentiable at** $x_0$ if

$$\exists\ f'(x_0) := \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \in \mathbb{R}.$$

The number $f'(x_0)$ is called **derivative of $f$ at** $x_0$.

**Definition 7.2.3: tangent line**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ as above, and let $f$ be differentiable at $x_0$. The straight line of equation

$$y = f(x_0) + f'(x_0)(x - x_0),$$

is called **tangent to** $f$ **at** $(x_0, f(x_0))$.

The first important remark that follows immediately from this definition is

**Proposition 7.2.4**

If $f$ is differentiable at $x_0$ then it is also continuous at $x_0$.

Proof. We have to show $\lim_{x \to x_0} f(x) = f(x_0)$ or, equivalently, $\lim_{x\to x_0}(f(x) - f(x_0)) = 0$. We have

$$\lim_{x\to x_0}(f(x) - f(x_0)) \overset{x=x_0+h}{=} \lim_{h\to 0}(f(x_0+h) - f(x_0)) = \lim_{h\to 0} \underbrace{\frac{f(x_0+h) - f(x_0)}{h}}_{\longrightarrow f'(x_0)} \cdot h = 0. \quad \square$$

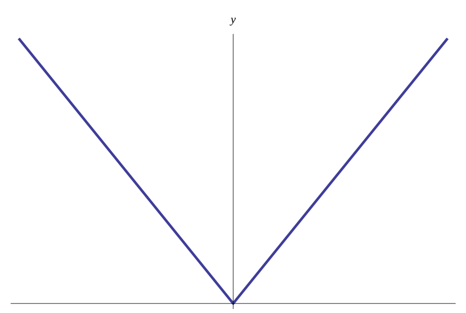
Remark 7.2.5 (Warning!). **There is always someone that believes that continuity implies differentiability and not vice versa. This is wrong!** A classical example is the function $f(x) := |x|$. It is easy to check that $\nexists f'(0)$.

Indeed,

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

This limit does not exist because

$$\lim_{h\to 0+} \frac{|h|}{h} = \lim_{h\to 0+} \frac{h}{h} = 1, \quad \lim_{h\to 0-} \frac{|h|}{h} = \lim_{h\to 0-} \frac{-h}{h} = -1. \quad \square$$




We introduced axiomatically the concept of *tangent line*. Since "tangent" is usually referred to some geometric idea, one may wonder in which geometrical sense the tangent line is actually tangent. To illustrate this point, let consider a generic (non-vertical) straight line passing through the point $(x_0, f(x_0))$, that is

$$y = m(x - x_0) + f(x_0).$$

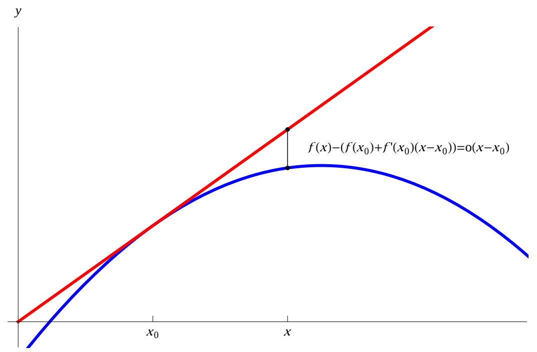
We define the gap between $f$ and this line as

$$\varepsilon_m(x) := f(x) - (m(x - x_0) + f(x_0)).$$

If $f$ is differentiable at $x_0$, it is also continuous, thus

$$\lim_{x\to x_0} \varepsilon_m(x) = \lim_{x\to x_0} (f(x) - (m(x - x_0) + f(x_0))) = 0,$$

that is, the gap $\varepsilon(x)$ vanishes at $x_0$. From this point of view, all straight lines through $(x_0, f(x_0))$ are the same.




What makes the tangent line special is that *for tangent line the gap $\varepsilon(x)$ goes to 0 faster than for any other line*. This may be interpreted as follows: *among all straight lines passing through the point $(x_0, f(x_0))$, the tangent is that one with the best approximation error.*

**Proposition 7.2.6**

Let $f$ be a differentiable function at $x_0$. Then

$$\lim_{x\to x_0} \frac{\varepsilon_m(x)}{x - x_0} = f'(x_0) - m.$$

In particular,

$$\varepsilon_m(x) = o(x - x_0), \quad \Longleftrightarrow \quad m = f'(x_0).$$

Proof.

$$\lim_{x\to x_0} \frac{\varepsilon_m(x)}{x - x_0} = \lim_{x\to x_0} \frac{f(x) - (m(x - x_0) + f(x_0))}{x - x_0} = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} - m = f'(x_0) - m.$$

□ As a straightforward corollary we get that differentiability at a point is equivalent to a certain approximation with a linear function:

Corollary 7.2.7. *Consider a function $f : D \to \mathbb{R}$ and let $x_0 \in \mathrm{Int}(D)$. $f$ is differentiable at $x_0$ if and only if*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0) \quad \textit{for} \;\; x \to x_0$$

*or, equivalently,* $f(x_0 + h) = f(x_0) + f'(x_0)h + o(h)$, *for* $h \to 0$.

Proof. Indeed, using the notation of the previous proposition one has

$$o(x - x_0)0\varepsilon_{f'(x_0)}(x) = f(x) - f'(x_0)(x - x_0) + f(x_0).$$

□

Remark 7.2.8. Using Corollary 7.2.7 one gets a faster proof of the implication *differentiable at $x_0$ $\Longrightarrow$ continuous at $x_0$*. Indeed,

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} f(x_0) + f'(x_0)(x - x_0) + o(x - x_0) = f(x_0).$$

To introduce the notion of left and right derivative

**Definition 7.2.9**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function, and let $[x_0, x_0 + \eta] \subseteq D$ $\left[\text{resp. } [x_0 - \eta, x_0] \subseteq D\right]$, for some $\eta > 0$. [a] We say that $f$ is **differentiable from the right at** $x_0$ $\left[\text{resp. } \textbf{differentiable from the left at } x_0\right]$ if

$$\exists \lim_{h \to 0+} \frac{f(x_0 + h) - f(x_0)}{h} =: f'_+(x_0) \in \mathbb{R}.$$

$$\left[\text{resp. } \exists \lim_{h \to 0-} \frac{f(x_0 - h) - f(x_0)}{h} =: f'_-(x_0) \in \mathbb{R}. \right]$$

[a] The intervals $[x_0, x_0 + \eta]$ and $[x_0 - \eta, x_0]$ are often called right and left neighbourhood of $x_0$, respectively

It is clear that

**Proposition 7.2.10**

A function $f$ is differentiable at $x_0$ iff it is differentiable from the left and from the right at $x_0$ and $f'_-(x_0) = f'_+(x_0) = f'(x_0)$.

### 7.3. Derivative of elementary functions

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$. We recall that $f'(x)$ can be defined only at points $x \in D$. Thus, set

$$D' := \{x \in D \ : \ \exists f'(x)\} \subset D,$$

is the domain of definition for $f'$. We stress that it might well happen that $D' \subsetneq D$. For example, if $f(x) = |x|$, $D = \mathbb{R}$ while $D' = \mathbb{R} \backslash \{0\} \subsetneq D$. Many elementary functions are differentiable in their natural domains apart, at most, at some exceptional point. Derivative of elementary functions follows from the fundamental limits of Section 5.5.

**Proposition 7.3.1**

We have

- Constant functions are differentiable on $\mathbb{R}$ with null derivative.
- $e^x$ is differentiable on $\mathbb{R}$ and $(e^x)' = e^x$.
- $\log x$ is differentiable on $]0,+\infty[$ and $\log'(x) = \frac{1}{x}$.
- $\sin x$ and $\cos x$ are differentiable $\mathbb{R}$ and $\sin'(x) = \cos x$ while $\cos'(x) = \sin x$.
- $\sinh x$ and $\cosh$ are differentiable $\mathbb{R}$ and $\sinh'(x) = \cosh x$ while $\cosh' = \sinh$.
- $x^n$, $n \in \mathbb{N}$ is differentiable on $\mathbb{R}$ and $(x^n)' = nx^{n-1}$.
- $x^m$, $m \in \mathbb{Z}$, $m < 0$ is differentiable on $\mathbb{R}\backslash\{0\}$ and $(x^m)' = mx^{m-1}$.
- $x^\alpha$, $\alpha \in \mathbb{R}$, is differentiable on $]0,+\infty[$ (and also at $0$ from the right if $\alpha \geqslant 1$) and $(x^\alpha)' = \alpha x^{\alpha-1}$.
- $|x|$ is differentiable on $\mathbb{R}\backslash\{0\}$ and $|x|' = \operatorname{sgn}(x)$.

Proof. Let $f(x) \equiv C$. Then

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{C-C}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Passing to the exponential,

$$\lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = e^x \lim_{h\to 0} \frac{e^h-1}{h} = e^x.$$

For the logarithm we have

$$\lim_{h\to 0} \frac{\log(x+h)-\log x}{h} = \lim_{h\to 0} \frac{\log\left(x\left(1+\frac{h}{x}\right)\right)-\log x}{h} = \lim_{h\to 0} \frac{\log\left(1+\frac{h}{x}\right)}{h/x}\frac{1}{x} = \frac{1}{x}.$$

Let's pass to trigonometric functions. For sine we have

$$\lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{\sin x\cos h+\sin h\cos x-\sin x}{h} = \lim_{h\to 0}\left(\sin x\frac{\cos h-1}{h^2}h+\cos x\frac{\sin h}{h}\right) = \cos x.$$

A similar computation shows that $\cos' = -\sin$. Hyperbolic functions sinh and cosh work similarly. For power, let us start with $n \in \mathbb{N}$:

$$\frac{(x+h)^n-x^n}{h} = \frac{1}{h}\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}h^k - x^n\right) = \frac{1}{h}\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^k = \sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^{k-1}.$$

As $k \geqslant 2$ we have $h^{k-1} \longrightarrow 0$ as $h \longrightarrow 0$. So

$$\lim_{h\to 0}\frac{(x+h)^n-x^n}{h} = \binom{n}{1}x^{n-1} = nx^{n-1}.$$

If $m \in \mathbb{Z}$, $m = -n$ with $n > 0$, then, for $x \neq 0$,

$$\frac{(x+h)^m - x^m}{h} = \frac{1}{h}\left(\frac{1}{(x+h)^n} - \frac{1}{x^n}\right) = \frac{1}{h}\frac{x^n - (x+h)^n}{(x+h)^n x^n} = \frac{1}{(x+h)^n x^n}\frac{x^n - (x+h)^n}{h}$$

$$\longrightarrow \frac{1}{x^{2n}}\left(-nx^{n-1}\right) = -nx^{-n-1} = mx^{m-1}.$$

More complex is the case of real exponent. Taking $x > 0$

$$\lim_{h\to 0}\frac{(x+h)^\alpha - x^\alpha}{h} = \lim_{h\to 0} x^\alpha \frac{\left(1+\frac{h}{x}\right)^\alpha - 1}{h} \overset{t=\frac{h}{x},\ h=tx}{=} x^\alpha \lim_{t\to 0}\frac{(1+t)^\alpha - 1}{tx} = x^{\alpha-1}\alpha.$$

If $x = 0$ and $\alpha > 1$ then the incremental ratio becomes

$$\lim_{h\to 0+}\frac{h^\alpha}{h} = \lim_{h\to 0+} h^{\alpha-1} = 0,$$

that confirm the rule $\alpha x^{\alpha-1}$ for $x = 0$. Finally, for modulus it is an easy exercise. □

Derivative of tan, arcsin, arccos, arctan, $\sinh^{-1}$ and $\cosh^{-1}$ will be computed later.

## 7.4. Rules of calculus

We need efficient rules to compute the derivative of functions composed either by algebraic operations or compositions between elementary functions. We start by algebraic rules:

**Proposition 7.4.1**

Let $f, g$ be differentiable at $x_0$. Then

i) $f \pm g$ is differentiable at $x_0$ and $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$.
ii) $f \cdot g$ is differentiable at $x_0$ and $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.
iii) if $g(x_0) \neq 0$ then $f/g$ is differentiable at $x_0$ and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

.

In particular, we have the formulas

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0), \quad \text{(linearity)}, \quad \left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}, \quad (\text{if } g(x_0) \neq 0).$$

Proof. The proof is easy. For the sum we have:

$$\frac{(f+g)(x_0+h) - (f+g)(x_0)}{h} = \frac{f(x_0+h) + g(x_0+h) - (f(x_0)+g(x_0))}{h}$$

$$= \frac{f(x_0+h) - f(x_0)}{h} + \frac{g(x_0+h) - g(x_0)}{h} \longrightarrow f'(x_0) + g'(x_0).$$

For the product we have

$$\begin{aligned}\frac{(f\cdot g)(x_0+h)-(f\cdot g)(x_0)}{h} &= \frac{f(x_0+h)g(x_0+h)-f(x_0)g(x_0)}{h}\\ &= \frac{f(x_0+h)-f(x_0)}{h}g(x_0+h)+f(x_0)\frac{g(x_0+h)-g(x_0)}{h}\\ &\longrightarrow f'(x_0)g(x_0)+f(x_0)g'(x_0).\end{aligned}$$

In the last passage we used $g(x_0+h)\longrightarrow g(x_0)$ because $g$ is continuous being differentiable. Similarly works the formula for the ratio. The particular cases are immediate by the general formulas. □

Let us pass now to the composition.

**Theorem 7.4.2: chain rule**

Assume that $\exists\ f'(x_0)$ and $\exists\ g'(f(x_0))$. Then

$$\exists\ (g\circ f)'(x_0) = g'(f(x_0))\,f'(x_0). \tag{7.4.1}$$

Proof. Since the map $y\mapsto g$ is differentiable at $f(x_0)$, by Corollary 7.2.7 one has

$$g(f(x)) = g(f(x_0)) + g'(f(x_0))(f(x)-f(x_0)) + o(f(x)-f(x_0))$$

Since also $f$ is differentiable at $x_0$, so that $f(x)-f(x_0) = f'(x_0)(x-x_0)+o(x-x_0)$, one gets

$$\begin{aligned}g(f(x)) &= g(f(x_0)) + g'(f(x_0))\big(f'(x_0)(x-x_0)+o(x-x_0)\big) + o\big(f'(x_0)(x-x_0)+o(x-x_0)\big)\\ &= g(f(x_0)) + g'(f(x_0))f'(x_0)(x-x_0)+o(x-x_0)\end{aligned}$$

(where we have made use of the trivial equality $o\big(f'(x_0)(x-x_0)+o(x-x_0)\big)\big) = o(x-x_0)$), so that

$$g(f(x)) = g(f(x_0)) + g'(f(x_0))f'(x_0)(x-x_0)+o(x-x_0),$$

which, in view of Corollary 7.2.7, yields

$$(g\circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

□

In particular, *composing functions differentiable in their domains produces a function differentiable on its domain*. However, we may have that the components are not differentiable while the composition it is.

Remark 7.4.3 (Warning!). Once more, this is a quite a common belief, probably caused by a logical misunderstanding. That is, the belief that $g\circ f$ is differentiable at $x_0$ **if and only if** $f$ is differentiable at $x_0$ and $g$ is differentiable at $f(x_0)$. Therefore, in particular, and this is **wrong**, $g\circ f$ cannot be differentiable at $x_0$ if both $f$ and $g$ are not differentiable at, resp., $x_0$ and $f(x_0)$. To show the mistake, consider the composition

$$x \overset{f}{\longmapsto} |x| \overset{g}{\longmapsto} |x|^2,\ \text{that is } f(x)=|x|,\ g(y)=y^2.$$

Then $g\circ f(x) = |x|^2 = x^2$ which is clearly differentiable on $\mathbb{R}$. Nevertheless, $f$ is not differentiable at 0.

The point is: the chain rule provides only a *sufficient condition* (that is, $\exists f'(x_0)$ and $\exists g'(f(x_0))$) ensuring $g \circ f$ is differentiable. If the premises are not verified, $g \circ f$ might still be differentiable. This has to be checked directly. □

EXAMPLE 7.4.4. *Discuss differentiability of the function* $|e^{x^3} - 1|$ *and compute the derivative at points of where the function is differentiable.*

SOL. — Let $f(x) = |e^{x^3} - 1|$, defined on $D = \mathbb{R}$. Since $|\cdot|$ is differentiable on $\mathbb{R}\backslash\{0\}$, exp and $x^3$ are both differentiable on $\mathbb{R}$, $f$ turns out to be the composition of differentiable functions for $x$ such that $e^{x^3} - 1 \neq 0$. Now, since

$$e^{x^3} - 1 = 0, \iff e^{x^3} = 1, \iff x^3 = 0, \iff x = 0.$$

we conclude that certainly $f$ is differentiable on $\mathbb{R}\backslash\{0\}$. What about $x = 0$? Can we say that $f$ is not differentiable at this point? As pointed out in the previous remark, this cannot be decided by the chain rule. We have to discuss directly applying the definition:

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|e^{h^3} - 1| - 0}{h} = \lim_{h \to 0} \left| \frac{e^{h^3} - 1}{h^3} \right| \frac{|h^3|}{h}.$$

Now, because of the fundamental limit for the exponential

$$\lim_{h \to 0} \frac{e^{h^3} - 1}{h^3} \overset{y=h^3}{=} \lim_{y \to 0} \frac{e^y - 1}{y} = 1,$$

while

$$\frac{|h|^3}{h} = h^2 \frac{|h|}{h} = h^2 \mathrm{sgn}(h) \longrightarrow 0.$$

We conclude $\exists f'(0) = 0$. Conclusion: $f$ is differentiable on $D' = \mathbb{R} = D$. To compute the derivative, we apply the chain rule for $x \neq 0$ ($f'(0) = 0$ has been already computed). Recalling that $|\cdot|' = \mathrm{sgn}$, we have

$$\left(|e^{x^3} - 1|\right)' = \mathrm{sgn}\left(e^{x^3} - 1\right)\left(e^{x^3} - 1\right)' = \mathrm{sgn}\left(e^{x^3} - 1\right)\left(e^{x^3}(x^3)' - 0\right) = \mathrm{sgn}\left(e^{x^3} - 1\right) 3x^2 e^{x^3}.$$

Finally, we may notice that $e^{x^3} - 1 > 0$ iff $e^{x^3} > 1$, iff $x^3 > 0$, iff $x > 0$. Thus $\mathrm{sgn}\left(e^{x^3} - 1\right) = \mathrm{sgn} x$ and we have

$$f'(x) = (\mathrm{sgn} x) 2x^2 e^{x^3}. \ \forall x \neq 0.$$

Even if sgn0 does not make sense, we may abuse a bit with the notation and take previous formula valid also for $x = 0$. □

## 7.5. Fundamental theorems of Differential Calculus

In this section, we will present the most important theorems of Differential Calculus: Fermat, Rolle, Lagrange, and Cauchy. Actually, the last three are corollaries of the first one. It was Fermat to notice that the minimum/maximum points for a function are, in certain circumstances, points where $f'(x) = 0$. Actually, this happens only if the extrema lies in the interior of the domain and it holds also for *local* minimum/maximum points, that is, points that are min/max on a part of the domain. Let us first introduce this concept:

**Definition 7.5.1**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$. A point $x_{min} \in D$ is called **local/relative minimum** for $f$ on $D$ if

$$\exists\, I_{x_{min}} \;:\; f(x_{min}) \leqslant f(x),\ \forall x \in D \cap I_{x_{min}}.$$

Similarly is defined a **local/relative maximum**. Local min/max are called **local extreme points**.

Of course, a global extreme point is also local, but not vice versa.

**Theorem 7.5.2: Fermat**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $x_0 \in \mathrm{Int}(D)$ be a local min/max point. If $f$ is differentiable at $x_0$ then $f'(x_0) = 0$. A point where $f'(x_0) = 0$ is called **stationary point for** $f$.

Proof. Consider the case of a local minimum $x_{min} \in \mathrm{Int}(D)$.

$$\exists\, I_{x_{min}} = [x_{min} - r, x_{min} + r] \;:\; f(x_{min}) \leqslant f(x),\ \forall x \in D \cap I_{x_{min}}.$$

Since $x_0 \in \mathrm{Int}(D)$, we may always assume $I_{x_{min}} \subset D$ (otherwise we reduce the size of $I_{x_{min}}$). In particular, Then,

$$\text{if } |h| < r,\ \Longrightarrow\ f(x_{min} + h) \geqslant f(x_{min}),\ \Longleftrightarrow\ f(x_{min} + h) - f(x_{min}) \geqslant 0.$$

Therefore

$$\left.\begin{aligned} f'(x_{min}) = f'_-(x_{min}) = \lim_{h \to 0-} \frac{f(x_{min} + h) - f(x_{min})}{h} \geqslant 0, \\ f'(x_{min}) = f'_+(x_{min}) = \lim_{h \to 0+} \frac{f(x_{min} + h) - f(x_{min})}{h} \leqslant 0, \end{aligned}\right\} \Longrightarrow\ f'(x_{min}) = 0. \qquad \square$$

Remark 7.5.3. Here some useful remarks on Fermat's theorem:

- **conclusion might be false if** $x_0 \notin \mathrm{Int}D$. For instance: if $D = [a, b]$ and we have a minimum at $x = a$ then $f'_+(a) \geqslant 0$, while if the minimum is at $x = b$ we have $f'_-(b) \leqslant 0$. This can be easily deduced by the previous proof. However, these values are not necessarily $= 0$: take $f(x) = x$, $x \in [0, 1]$. Clearly, $x = 0$ is a global minimum an $x = 1$ is a global maximum, and $f'(x) \equiv 1$ is never $= 0$.
- Fermat's theorem says that any local extreme point in the interior of the domain of a differentiable function is a stationary point. **Vice versa does not hold!** Take $f(x) = x^3$: $f \nearrow$ strictly, we have $f'(x) = 3x^2$ so $f'(0) = 0$, but 0 is neither a local minimum nor a local maximum.
- **it may happen that** $f$ **has min/max at some** $x_0 \in \mathrm{Int}D$ **but** $f'$ **does not exist**. Just think to $f(x) = |x|$: $x = 0$ is clearly a global minimum point but $f$ is not differentiable at $x = 0$. $\square$

Rolle's theorem expresses an intuitive fact: *if a smooth function $f$ on an interval has equal values at the extremes, then somewhere there's a point with horizontal tangent.*

**Theorem 7.5.4: Rolle**

Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $]a, b[$ [a]. Then

$$\text{if } f(a) = f(b), \implies \exists \xi \in ]a, b[: f'(\xi) = 0.$$

[a]This means: it is not required that $f$ be differentiable at $a$ and $b$.

Proof. Since $f \in \mathscr{C}([a, b])$, according to Weierstrass' theorem there are both global min and max on $[a, b]$. Let us call $x_{min}$ and $x_{max}$, respectively, global min (max) point for $f$ on $[a, b]$. There are two cases:

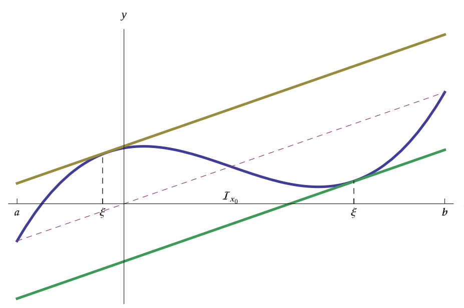
- $f(x_{min}) = f(x_{max})$: but then $f$ is constant, therefore $f' \equiv 0$;
- $f(x_{min}) < f(x_{max})$: in particular, one of the two must be in $]a, b[$ (otherwise, being $f(a) = f(b)$, we would be in the previous case). Hence, we have an extreme point in the interior of $[a, b]$, according to Fermat's theorem $f' = 0$ at that point.

In any case, we find at least an interior point where $f' = 0$. □

Consider now a differentiable function $f : [a, b] \longrightarrow \mathbb{R}$ and consider the chord joining initial and final points of the graph, that is $(a, f(a))$ and $(b, f(b))$. The chord slope is

$$\frac{f(b) - f(a)}{b - a}.$$

The figure suggests that, in general, at least one tangent to $f$ is parallel to the chord. This is the conclusion of the




**Theorem 7.5.5: Lagrange**

Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $]a, b[$. Then

$$\exists\, \xi \in ]a, b[: \frac{f(b) - f(a)}{b - a} = f'(\xi). \tag{7.5.1}$$

Equivalently:

$$\exists\, \xi \in ]a, b[: f(b) - f(a) = f'(\xi)(b - a). \tag{7.5.2}$$

This last is called **finite increment formula**.

Proof. It's just a consequence of the Rolle theorem. Indeed, consider the auxiliary function

$$h : [a, b] \longrightarrow \mathbb{R},\ h(x) := f(x) - \frac{f(b) - f(a)}{b - a}(x - a),\ x \in [a, b].$$

Clearly $h$ is continuous on $[a, b]$ and differentiable on $]a, b[$. Moreover

$$h(a) = f(a), \quad h(b) = f(b) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - (f(b) - f(a)) = f(a).$$

Therefore, by Rolle thm, there exists $\xi \in ]a, b[$ such that $h'(\xi) = 0$. But

$$0 = h'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{b - a}, \iff \frac{f(b) - f(a)}{b - a} = f'(\xi). \quad \square$$

An interesting consequence of the Lagrange formula is the following test of differentiability:

**Corollary 7.5.6**

Let $f : D \subset \mathbb{R} \longrightarrow R$ be a function and let $x_0 \in \mathrm{Int}(D)$ be such that

i) $f$ is continuous at $x_0$

ii) $f$ is differentiable on $I_{x_0} \setminus \{x_0\}$ for some neighbourhood $I_{x_0}$ of $x_0$.

Then

$$\exists \lim_{x \to x_0} f'(x) =: \ell \in \mathbb{R} \cup \{\pm\infty\}, \implies \begin{cases} \ell \in \mathbb{R}, \implies \exists f'(x_0) = \ell. \\ \\ \ell = \pm\infty, \implies \nexists f'(x_0), \end{cases}$$

Applying Lagrange's theorem on the interval $[x_0, x_0 + h]$, there exists $\xi_h \in ]x_0, x_0 + h[$ such that

$$\frac{f(x_0 + h) - f(x_0)}{h} = f'(\xi_h).$$

Being $x_0 < \xi_h < x_0 + h$ we have $\xi_h \longrightarrow x_0+$ (two policemen thm), so, by hypothesis, $f'(\xi_h) \longrightarrow \ell$. Therefore

$$f'_+(x_0) = \lim_{h \to 0+} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0+} f'(\xi_h) = \ell.$$

Similarly, by applying Lagrange's theorem on the interval $[x_0 - h, x_0]$ one gets

$$f'_-(x_0) = \ell,$$

so that $f'(x_0) = \ell$. as soon as $\ell \in \mathbb{R}$. $\square$

Notice that from the proof of Corollary 7.4 one actually gets the following stronger result:

**Corollary 7.5.7**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function, and let $[x_0, x_0+\eta] \subseteq D$ $\big[$resp. $[x_0-\eta, x_0] \subseteq D\big]$, for some $\eta > 0$.

i) $f$ is right [resp. left] continuous at $x_0$

ii) $f$ is differentiable on $]x_0, x_0+\eta]$ $\big[$resp. on $[x_0-\eta, x_0[\ \big]$, for some $\eta > 0$

Then

$$\exists \lim_{x\to x_0+} f'(x) =: \ell \in \mathbb{R}\cup\{\pm\infty\}, \implies \begin{cases} \ell \in \mathbb{R}, \implies \exists f'_+(x_0) = \ell. \\ \\ \ell = \pm\infty, \implies \nexists f' + (x_0), \end{cases}$$

$$\left[\text{risp.}\exists \lim_{x\to x_0-} f'(x) =: \ell \in \mathbb{R}\cup\{\pm\infty\}, \implies \begin{cases} \ell \in \mathbb{R}, \implies \exists f'_-(x_0) = \ell \\ \\ \ell = \pm\infty, \implies \nexists f'_-(x_0) \end{cases}\right]$$

Remark 7.5.8 (Warning!). **Nothing can be said the limit** $\lim_{x\to x_0+} f'(x)$ **does not exist**. For example, let

$$f(x) := \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0, \\ \\ 0, & x = 0. \end{cases}$$

Easily we have $f$ continuous at $x = 0$. Moreover

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h\to 0} h \sin \frac{1}{h} = 0,$$

However,

$$\lim_{x\to 0+} f'(x) = \lim_{x\to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x}\right) = -\lim_{x\to 0+} \cos \frac{1}{x},$$

does not exist! $\square$

Example 7.5.9. *Let* $f : \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ *be defined as*

$$f(x) := \begin{cases} a \arctan \frac{1}{x} + (b+1)\log(1-x), & x < 0, \\ \\ \sinh \frac{bx}{x^2+1} - a \cos \frac{1}{\sqrt{x}}, & x > 0. \end{cases}$$

*Are there values* $a, b \in \mathbb{R}$ *such that* $f$ *extends continuously at* $x = 0$*? Is such an extension differentiable at* $x = 0$*?*

Sol. — $f$ extends continuously at $x = 0$ iff $\exists \lim_{x\to 0} f(x)$. We have

$$f(0-) = \lim_{x\to 0-} \left(a \arctan \frac{1}{x} + (b+1)\log(1-x)\right) = -a\frac{\pi}{2},$$

while, since $\lim_{x\to 0+} \sinh \frac{bx}{x^2+1} = 0$, we have

$$\exists \lim_{x\to 0+} \left(\sinh \frac{bx}{x^2+1} - a \cos \frac{1}{\sqrt{x}}\right), \iff a = 0.$$

In case $a = 0$, $\lim_{x\to 0+} f(x) = 0$ thus $\lim_{x\to 0} f(x) = 0$. In other words, the unique possible extension is

$$f(x) = \begin{cases} (b+1)\log(1-x), & x < 0, \\ \\ 0, & x = 0, \\ \\ \sinh \frac{bx}{x^2+1}, & x > 0, \end{cases}$$

For $x \neq 0$ we have

$$f'(x) = \begin{cases} -\frac{b+1}{1-x}, & x < 0, \\ \\ \left(\cosh \frac{bx}{x^2+1}\right) \frac{b(x^2+1)-2bx^2}{(x^2+1)^2}, & x > 0. \end{cases}$$

By this it follows that

$$\lim_{X\to 0-} f'(x) = \lim_{x\to 0-} -\frac{b+1}{1-x} = -b-1, \quad \lim_{x\to 0+} f'(x) = \lim_{x\to 0+} \left(\cosh \frac{bx}{x^2+1}\right) \frac{b(x^2+1)-2bx^2}{(x^2+1)^2} = b.$$

Applying the test of differentiability, $\exists f'_-(0) = -b-1$ and $\exists f'_+(0) = b$. But then $\exists f'(0)$ iff $f'_-(0) = f_+(0)$, that is $-b-1 = b$, or $b = -\frac{1}{2}$. □

The following theorem, due to Cauchy, is a generalization of the Lagrange Theorem:

**Theorem 7.5.10: Cauchy**

Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $]a,b[$, with $g(a) \neq g(b)$ and $g'(x) \neq 0$ for any $x \in ]a,b[$. Then

$$\exists\, \xi \in ]a,b[\, : \, \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}. \tag{7.5.3}$$

## 7.6. Derivative and monotonicity

In this section, we establish the fundamental connection between derivative and monotonicity. This is the crucial step in the method for searching min/max of a function $f$.

**Theorem 7.6.1**

Let $I$ be an interval and let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, differentiable on $\mathrm{Int}(I)$. Then

$$f \nearrow \ (f \searrow) \text{ on } I \iff f' \geqslant 0, \ (f' \leqslant 0) \text{ on } I.$$

Moreover, if $f' > 0$ (or $f' < 0$) on $I$ the monotonicity is strict.

Proof. We will deal only with the part concerning $f$ increasing, the part on $f$ decreasing being perfectly analogous (do it by exercise). Let us prove the implication " $\Longrightarrow$ ". Suppose $f \nearrow$. Then, if $x \in \mathrm{Int}(I)$,

$$f(x+h) \geqslant f(x), \ \forall h > 0, \implies f'(x) = f'_+(x) = \lim_{h\to 0+} \frac{f(x+h)-f(x)}{h} \geqslant 0,$$

by permanence of sign.

Let us prove the implication " $\Longleftarrow$ " Suppose $f' \geqslant 0$ on $I$ and take $x, y \in I$, $x < y$. The interval $[x, y] \subset I$ (because $I$ is an interval). Being $f$ differentiable on $I$ it is continuous. In other words, $f$ is continuous on $[x, y]$ and differentiable on $]x, y[$: applying Lagrange's finite increment formula (7.5.2) on $[x, y]$, there exists $\xi \in ]x, y[$ such that

$$f(y) - f(x) = f'(\xi)(y - x) \geqslant 0.$$

This means $f(x) \leqslant f(y)$ and because $x \leqslant y$ are arbitrary, we proved $f \nearrow$. If $f' > 0$ on $I$ then $f'(\xi) > 0$ and we obtain $f(x) < f(y)$, that is $f \nearrow$ strictly. $\square$

Remark 7.6.2 (Warning!). *Be careful! $f$ may be strictly increasing but $f' \not> 0$.* For instance: take $f(x) = x^3$. This is a clearly strictly increasing function on $\mathbb{R}$ but $f'(0) = 0$. $\square$

In particular, we have the

**Corollary 7.6.3**

Let $I$ be an interval and let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, differentiable on $\mathrm{Int}(I)$, and let $x_0 \in I$ be such that

i) $\exists f'(x) \leqslant 0$ [resp. $\exists f'(x) \geqslant 0$ ], $x \in I \cap ]-\infty, x_0[$;
ii) $\exists f'(x) \geqslant 0$ [resp. $\exists f'(x) \geqslant 0$ ], $x \in I \cap ]x_0, \infty[$;
iii) $f$ is continuous at $x_0$.l

Then $x_0$ is a minimum [resp. maximum] point for $f$ on $I$.

Proof. Let us prove just the part concerning the minimum, the part on the maximum being completely analogous. If $f' \leqslant 0$ on $I \cap ]-\infty, x_0[$ (which is of course an interval) then, by previous thm, $f \searrow$ on it. Therefore

$$f(x) \geqslant f(y), \ \forall\, x < y < x_0, \implies f(x) \geqslant \lim_{y \to x_0-} f(x) = f(x_0), \text{ by continuity at } x_0.$$

Similarly, $f(x) \geqslant f(x_0)$ as $x > x_0$. We conclude that

$$f(x) \geqslant f(x_0), \ \forall x \in I,$$

that is $x_0$ is a minimum point for $f$ on $I$. $\square$

Remark 7.6.4. *It is not required that $\exists f'(x_0)$.* This is to include cases where the function has an **angle point** (like modulus at $x = 0$) or a **cusp** (like the function $\sqrt{|x|}$ at $x = 0$) that are clearly minimum (or maximum in other examples) points. Of course: if $\exists f'(x_0)$, never mind!

Example 7.6.5. *Determine min/max (if any) of $f(x) = xe^{-x}$ on $D = [0, +\infty[$.*

Sol. — We notice first that $f$ is well defined, continuous and derivable on $D$. Clearly, $f(0) = 0$ and $f(x) \geqslant 0$ for every $x \in [0, +\infty[$. Since $D$ is not a closed and bounded interval, Weierstrass' theorem does not apply. However, we have

$$f'(x) = e^{-x} - xe^{-x} = (1 - x)e^{-x},$$

so

$$f'(x) \geqslant 0, \iff 1 - x \geqslant 0, \iff x \leqslant 1.$$

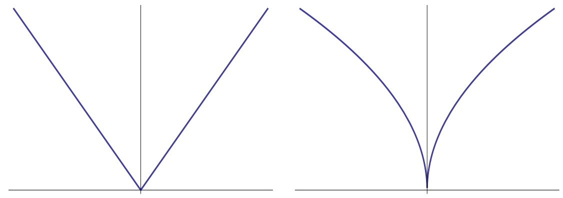
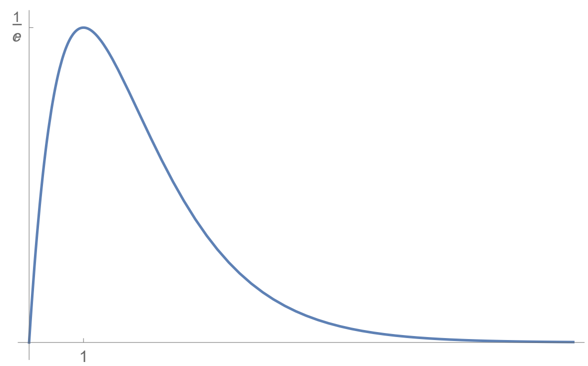


FIGURE 1. Angle point (left) and cusp (right).

Thus $f \nearrow$ on $[0,1]$, $f \searrow$ on $[1,+\infty[$ and since $f$ is also continuous at $x=1$ we conclude that $x=1$ is a global maximum point for $f$ on $D$. The same argument tells that $x=0$ is a minimum for $f$ on $[0,1]$, thus it is a local minimum for $f$. Since $f(x)>0$ for $x>0$ and $f(0)=0$ clearly $f(0)\leqslant f(x)$ for every $x\in D$, thus $x=0$ is a global minimum. There are no other min/max points. $\square$

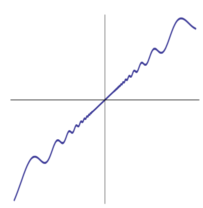



A final remark. At 99% of times, intuition works. However, there's still that 1% of subtleties which makes our apparently clear theory something at all trivial. The question is the following: suppose that $f'(x_0)>0$. The intuitions suggest that $f \nearrow$ at least on a small neighborhood of $x_0$. This is false!

EXAMPLE 7.6.6. *Let*

$$f(x) := \begin{cases} x+2x^2\cos\frac{1}{x}, & x\neq 0, \\ \\ 0, & x=0. \end{cases}$$

*Check that* $\exists f'(0)=1$ *but* $f$ *is not monotone in any neighborhood of* $0$.



SOL. — Easily $f\in\mathscr{C}(\mathbb{R})$ and differentiable clearly as $x\neq 0$. For $x=0$ we may notice that

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\left(1+2h\cos\frac{1}{h}\right)=1.$$

As $x\neq 0$,

$$f'(x)=1+4x\cos\frac{1}{x}-2\sin\frac{1}{x}.$$

Here you notice that as $x \longrightarrow 0$, $4x \cos \frac{1}{x} \longrightarrow 0$ while $2 \sin \frac{1}{x}$ oscillates between $-2$ and $2$ so, reasonably, $f'$ will assume always positive and negative values when $x$ is arbitrarliy close to 0. For instance

$$f'\left(\frac{1}{\frac{\pi}{2}+k\pi}\right) = 1 - 2\sin\left(\frac{\pi}{2}+k\pi\right) = 1 + 2(-1)^k = \begin{cases} 3, & k \text{ even,} \\ -1, & k \text{ odd.} \end{cases}$$

This shows that we cannot find an $I_0$ where $f \nearrow$, because otherwise it should be $f' \geqslant 0$ in $I_0$ but points $\frac{1}{\frac{\pi}{2}+k\pi}$ with $k$ odd belongs to any $I_0$ when $k$ is big enough and in those points $f' < 0$. $\square$

### 7.7. Inverse mapping theorem

Recall that *a strictly monotone bijective function $f : I \longrightarrow J$ with $I, J$ intervals is continuous with a continuous inverse*. We want now to replace *continuous* with *differentiable*. Let us first introduce an important class of functions:

**Definition 7.7.1**

We say that $f \in \mathscr{C}^1(D)$ if $f, f' \in \mathscr{C}(D)$.

We have the

**Theorem 7.7.2**

Let $f \in \mathscr{C}^1(I)$ on $I$ interval with $f' > 0$ or $f' < 0$ on $I$. Then $f$ is invertible between $I$ and $J := f(I)$, $f^{-1} \in \mathscr{C}^1(\mathbb{R})$ and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}, \ \forall y \in J. \tag{7.7.1}$$

Proof. By assumptions we derive $f \nearrow$ or $f \searrow$ strictly, thus $f : I \longrightarrow f(I) =: J$ is strictly monotone. $I$ is an interval, also $J$ it is because of the intermediate value theorem. As consequence, $\exists f^{-1} \in \mathscr{C}(J)$. We need to check that $f^{-1}$ is differentiable, that is to compute

$$\lim_{h\to 0} \frac{f^{-1}(y+h) - f^{-1}(y)}{h}.$$

Setting

$$x := f^{-1}(y),\ x + k := f^{-1}(y+h), \iff k = f^{-1}(y+h) - f^{-1}(y) \longrightarrow 0, \text{ as } h \longrightarrow 0$$

being $f^{-1}$ continuous. Thus, changing variable in the limit we get

$$(f^{-1})'(y) = \lim_{h\to 0} \frac{f^{-1}(y+h) - f^{-1}(y)}{h} = \lim_{k\to 0} \frac{k}{f(x+k) - f(x)} = \lim_{k\to 0} \frac{1}{\frac{f(x+k)-f(x)}{k}} = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}.$$

Finally, since $f', f^{-1}$ are continuous, $f' \circ f^{-1} \in \mathscr{C}(J)$ (composition of continuous functions) and because $f' \neq 0$ by hypothesis, it follows immediately that $(f^{-1})' \in \mathscr{C}(J)$, so $f^{-1} \in \mathscr{C}^1(J)$. $\square$

EXAMPLE 7.7.3 (arcsin, arccos). We already defined

$$\arcsin = \sin^{-1} : [-1,1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

and $\arcsin \in \mathscr{C}([-1,1])$. Now, since $f(x) = \sin x$ has $f'(x) = \cos x > 0$ for $x \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$. According to the previous theorem,

$$\exists \arcsin'(y) = \frac{1}{\sin'(\arcsin y)} = \frac{1}{\cos(\arcsin y)},\ \forall y \in ]-1,1[.$$

Then, by the fundamental identity $\cos^2 + \sin^2 = 1$ we have

$$\cos(\arcsin y) = \pm\sqrt{1-(\sin(\arcsin y))^2} = \pm\sqrt{1-y^2},$$

and since $\arcsin y \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$, $\cos(\arcsin y) > 0$ thus the correct sign is $+$. Thus

$$\arcsin' y = \frac{1}{\sqrt{1-y^2}},\ \forall y \in ]-1,1[.$$

Similarly, we proceed with arccos. The final result is

$$\arccos'(y) = -\frac{1}{\sqrt{1-y^2}},\ \forall y \in ]-1,1[. \quad \square$$

EXAMPLE 7.7.4 (arctan). Consider $\tan := \frac{\sin}{\cos} : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[ \longrightarrow \mathbb{R}$. On such interval $\tan \in \mathscr{C}^1$ and being

$$\tan'(x) = \frac{\cos x \cos x + \sin x \sin x}{(\cos x)^2} = \frac{1}{(\cos x)^2} \equiv 1 + (\tan x)^2 > 0,\ \forall x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[,$$

we can apply the differentiable inverse mapping theorem: $\arctan := \tan^{-1} \in \mathscr{C}^1$ and

$$\arctan'(y) = \frac{1}{\tan'(\arctan y)} = \frac{1}{1+(\tan(\arctan y))^2} = \frac{1}{1+y^2},\ \forall y \in \mathbb{R}. \quad \square$$

EXAMPLE 7.7.5 ($\sinh^{-1}$, $\cosh^{-1}$). Recall that $\sinh : \mathbb{R} \longrightarrow \mathbb{R}$ has inverse continuous $\sinh^{-1}$. Since $\sinh' = \cosh > 0$ thus inverse mapping theorem applies: we deduce $\sinh^{-1} \in \mathscr{C}^1$ and

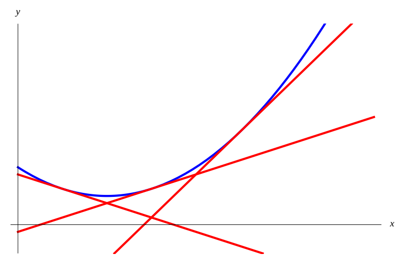
$$(\sinh^{-1})'(y) = \frac{1}{\sinh'(\sinh^{-1} y)} = \frac{1}{\cosh(\sinh^{-1} y)} = \frac{1}{\sqrt{(\sinh(\sinh^{-1} y))^2 + 1}} = \frac{1}{\sqrt{y^2+1}}.$$

Similarly,

$$(\cosh^{-1})'(y) = \frac{1}{\sqrt{y^2-1}}. \quad \square$$

## 7.8. Convexity

There is another important property that we may recognize by looking at the graph of a function $f$: its *curvature*. Roughly speaking, we may distinguish two types of curvature: upward and downward, according the graph of $f$ looks as a "parabola" oriented upward or downward. To identify a precise condition, let us consider the case of a regular function (that is, with tangent at every point) with upward curvature as in the figure.




Plotting some tangents, we may notice this phenomenon: *the graph of the function lies above all of its tangents*. This becomes the starting point of a definition:

Definition 7.8.1. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable on $I$ interval. We say that $f$ is* **convex** *on $I$ if*

$$f(x) \geqslant f(x_0) + f'(x_0)(x - x_0), \ \forall x \in I, \ \forall x_0 \in I. \tag{7.8.1}$$

*We say that $f$ is* **concave** *on $I$ if $-f$ is convex on $I$.*

Notice that the role of $x$ and $x_0$ in (7.8.1) is specular, thus we will conveniently rewrite it as

$$f(x) \geqslant f(y) + f'(y)(x - y), \ \forall x, y \in I. \tag{7.8.2}$$

To check (7.8.2) seems a nontrivial task, so we look for some equivalent but more practical conditions to check. Looking at the previous figure, we may notice that, when we move from left to right, the straight line ”rotates” counter clockwise. This geometrical operation corresponds to increasing the angular coefficient. This leads to the following guess: *$f$ is convex on $I$ iff $f' \nearrow$ on $I$*. This turns out to be true:

**Theorem 7.8.2**

Let $f : I \longrightarrow \mathbb{R}$, $I$ interval, $f$ differentiable on $I$. Then

$$f \text{ is convex on } I, \iff f' \nearrow \text{ on } I.$$

Proof. $\Longrightarrow$ Assume $f$ convex, that is (7.8.2) holds. Fix $x < y$ and notice that

$$f(y) \geqslant f(x) + f'(x)(y - x), \overset{y-x>0}{\Longrightarrow} f'(x) \leqslant \frac{f(y) - f(x)}{y - x}.$$

Exchanging the role of $x$ and $y$,

$$f(x) \geqslant f(y) + f'(y)(x - y), \overset{x-y<0}{\Longrightarrow} f'(y) \geqslant \frac{f(x) - f(y)}{x - y} = \frac{f(y) - f(x)}{y - x} \geqslant f'(x),$$

by which $f'(x) \leqslant f'(y)$ that is $f' \nearrow$.

$\Longleftarrow$ Assume $f' \nearrow$ and let's prove (7.8.2). Assume, for instance, $x < y$ (the other case being similar): then, by Lagrange's theorem applied on $[x, y]$ we have

$$\exists \xi \in ]x, y[ : \frac{f(y) - f(x)}{y - x} = f'(\xi).$$

Now. since $f' \nearrow$ we have $f'(x) \leqslant f'(\xi) \leqslant f'(y)$, that is

$$f'(x) \leqslant \frac{f(y) - f(x)}{y - x} \leqslant f'(y),$$

by which (7.8.2) easily follows.

To finish, we have to prove the second statement. For this, we just notice that if $f'$ is derivable, then $f' \nearrow$ iff $(f')' \geqslant 0$. □

To check $f' \nearrow$ we may use again Differential Calculus: if $f'$ is derivable, we may check if $(f')' \geqslant 0$.

**Definition 7.8.3**

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable function such that $f'$ is differentiable at $x_0$. In this case, we say that $f$ is **two times differentiable at** $x_0$ and we pose

$$f''(x_0) := (f')'(x_0).$$

Therefore, if $f$ is two times differentiable on $I$,

$$f \text{ is convex on } I, \iff f'' \geqslant 0 \text{ on } I.$$

We close this section with a simple result that illustrates the role of convexity in optimization. In general, as we know, stationary points are not necessarily extreme points. However, if we know that the function is concave/convex, this becomes true:

**Proposition 7.8.4**

Let $f : I \longrightarrow \mathbb{R}$, $I$ interval, $f$ convex on $I$. A stationary point for $f$ is necessarily a minimum on $I$.

Proof. Assume $f'(x_0) = 0$. Since $f$ is convex,

$$f(x) \geqslant f(x_0) + f'(x_0)(x - x_0) = f(x_0), \ \forall x \in I. \quad □$$

This type of result, extended and enriched, is at the base of the solution to many economic optimization problems.

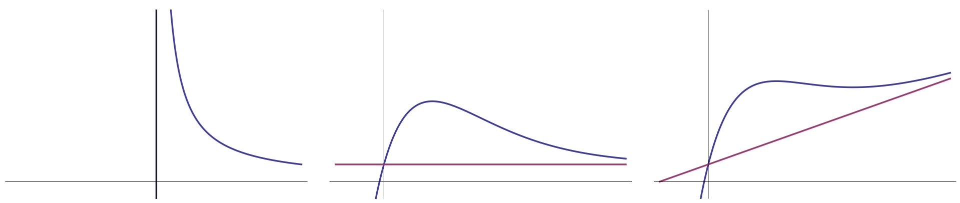
## 7.9. Plotting the graph of a function

We take now a break from the theory and show the application of the results seen so far to the problem of plotting the graph of a function. This problem may be used to discuss optimization problems as well as to discuss nontrivial inequalities (see below for an example).

The plot of the graph of a function follows after gathering several different informations, normally obtained by solving specific problems such as solving inequalities, computing limits and derivatives, and interpreting in a coherent way the informations. In general, we may group these informations into three blocks:

- *preliminaries* — domain of the function, its sign, its behaviour at the endpoints of the domain (limits and *asymptotes*, see below), continuity and eventual points where the function can be extended by continuity;
- *first derivative* — differentiability, behaviour of the derivative at the endpoints of its domain, sign of the derivative, monotonicity of the function, and local/global extreme points (if any);
- *second derivative* — differentiability of $f'$, sign of $f''$, convexity/concavity.

*Asymptotes* are straight lines that resemble the function in certain conditions. There are three types of asymptotes: *, vertical, horizontal, and oblique*:



The first one occurs when

$$\lim_{x\to x_0} f(x) = \pm\infty, \quad ,$$

or

$$\lim_{x\to x_0+} f(x) = \pm\infty, \quad \text{or} \quad \lim_{x\to x_0-} f(x) = \pm\infty.$$

In this case, $f$ becomes vertical at $x_0$, similarly to $x = x_0$. We say that $x = x_0$ is a **vertical asymptote** for $f$. The other two, horizontal and oblique asymptote, occurs at infinity. Suppose that

$$\lim_{x\to\pm\infty} f(x) = \ell \in \mathbb{R},$$

then $f$ becomes horizontal at $\pm\infty$, like $y = \ell$ at $\pm\infty$. We say that $y = \ell$ is an **horizontal asymptote**.

Finally, assume that

$$\lim_{x\to\pm\infty} f(x) = \pm\infty.$$

In this case, we might have that $f$ looks like $y = mx + q$ with $m \in \mathbb{R}\backslash\{0\}$ and $q \in \mathbb{R}$. For a formal definition, we say that $y = mx + q$ is an **oblique asymptote** for $f$ at $+\infty$ if

$$\lim_{x\to+\infty} (f(x) - (mx+q)) = 0.$$

The first problem is how to determine $m$ and $q$. The first remark is that, once $m$ has been determined,

$$q = \lim_{x\to+\infty} (f(x) - mx).$$

About $m$, notice that

$$\lim_{x\to\pm\infty} (f(x) - (mx+q)) = 0, \quad \Longrightarrow \quad \lim_{x\to\pm\infty} \frac{f(x) - (mx+q)}{x} = 0.$$

By this, since $\frac{q}{x} \longrightarrow 0$, we obtain

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} - m = 0, \quad \Longleftrightarrow \quad m = \lim_{x\to\pm\infty} \frac{f(x)}{x}.$$

Such limit must be $\neq 0$: indeed, if it were $m = 0$ one would obtain the contradiction

$$0 = \lim_{x\to\pm\infty}(f(x) - (mx+q)) = \lim_{x\to\pm\infty}(f(x) - q) = \pm\infty$$

.

Example 7.9.1. *Given the function*

$$f(x) = \arctan(e^x - 1) + \log|e^x - 4|$$

*find its domain, the behaviour at the endpoints of the domain (limits and asymptotes), continuity, differentiability, monotonicity, extreme points (if any) and plot a qualitative graph.*

Sol. — *Domain:* clearly $D(f) = \{x \in \mathbb{R} \,:\, e^x - 4 \neq 0\} = \{x \neq \log 4\} = ]-\infty, \log 4[\cup]\log 4, +\infty[$.

*Limits and asymptotes:* We have to check the behaviour at $\pm\infty$ and $\log 4\pm$.

$$f(-\infty) = \lim_{x\to-\infty}(\arctan(e^x-1) + \log|e^x - 4|) = \arctan(-1) + \log|-4| = -\frac{\pi}{4} + \log 4,$$

by which $y = -\frac{\pi}{4} + \log 4$ is horizontal asymptote at $-\infty$. At $+\infty$ easily we have $f(+\infty) = +\infty$, so it there may be an oblique asymptote $y = mx + q$. We have

$$m = \lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{\log(e^x-1)}{x} \overset{y=e^x-1}{=} \lim_{y\to+\infty}\frac{\log y}{\log(y+1)} = 1.$$

$$q = \lim_{x\to+\infty}(f(x) - x) = \lim_{x\to+\infty}(\arctan(e^x-1) + \log|e^x-4| - x) = \frac{\pi}{2} + \lim_{x\to+\infty}\log\frac{e^x-4}{e^x} = \frac{\pi}{2}.$$

Therefore, $y = x + \frac{\pi}{2}$ is oblique asymptote $+\infty$. It remains the behaviour at $\log 4\pm$. Being $|e^x - 4| \longrightarrow 0+$ as $x \to \log 4$ we have immediately that

$$\lim_{x\to\log 4} f(x) = -\infty.$$

Therefore, $x = \log 4$ is vertical asymptote for $f$.

*Continuity and differentiability:* on $D(f)$, $f$ is a composition of continuous and differentiable functions, therefore it is continuous and differentiable. About $f'$ we have

$$\begin{aligned} f'(x) &= \frac{e^x}{1+(e^x-1)^2} + \frac{1}{|e^x-4|}\operatorname{sgn}(e^x-4)e^x = e^x\left(\frac{1}{1+(e^x-1)^2} + \frac{1}{e^x-4}\right) \\ &= e^x\frac{e^x-4+1+(e^x-1)^2}{(1+(e^x-1)^2)(e^x-4)} = e^x\frac{e^x-4+1+e^{2x}-2e^x+1}{(1+(e^x-1)^2)(e^x-4)} = e^x\frac{e^{2x}-e^x-2}{(1+(e^x-1)^2)(e^x-4)} \end{aligned}$$

There are no points where it is interesting to compute limits for $f'$.

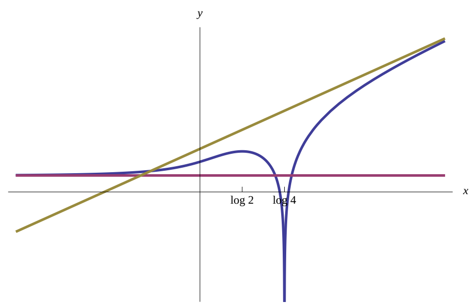
*Monotonicity:* We have

$$f'(x) \geqslant 0, \quad\Longleftrightarrow\quad \frac{e^{2x}-e^x-2}{e^x-4} \geqslant 0.$$

Now, setting $y = e^x$, we have $y^2 - y - 2 \geqslant 0$, iff $y \leqslant \frac{1-3}{2} = -1$ or $y \geqslant \frac{1+3}{2} = 2$, therefore iff $e^x \leqslant -1$ (never) or $e^x \geqslant 2$, that is $x \geqslant \log 2$. Moreover $e^x - 4 \geqslant 0$ iff $x \geqslant \log 4$. Therefore, we get the following table:

| | $-\infty \quad \log 2$ | $\log 2 \quad \log 4$ | $\log 4 \quad +\infty$ |
|---|---|---|---|
| $\mathrm{sgn}(e^{2x} - e^x - 2)$ | - | + | + |
| $\mathrm{sgn}(e^x - 4)$ | - | - | + |
| $\mathrm{sgn}(f')$ | + | $-$ | + |
| $f$ | $\nearrow$ | $\searrow$ | $\nearrow$ |

Being $f$ continuous at $x = \log 2$ we deduce that this is a maximum on $]-\infty, \log 4[$. Because $f$ is upper unbounded, the point is only a local maximum. There are not minimum points. We may conclude with the following graph. □




Example 7.9.2. *Let*

$$f(x) = \frac{x \log x}{(\log x - 1)^2}.$$

*Determine: the domain of $f$, sign, the behaviour at the endpoints of the domain (limits and asymptotes), continuity and differentiability, points where $f$ can be extended by continuity/differentiability, monotonicity and extreme points. Finally, plot a qualitative graph.*

Sol. — *Domain:* Clearly $D(f) = \{x \in \mathbb{R} \,:\, x > 0,\ \log x \neq 1\} = \{x \in \mathbb{R} \,:\, x > 0,\ x \neq e\} = ]0, e[\cup]e, +\infty[$.

*Sign:* We have

$$f \geqslant 0, \iff \log x \geqslant 0, \iff x \geqslant 1,$$

and $f = 0$ iff $\log x = 0$, that is $x = 1$.

*Limits and asymptotes:* We have to check the behaviour of $f$ at $0+$, $e\pm$ and $+\infty$. We have

$$f(0+) = \lim_{x \to 0+} \frac{x \log x}{(\log x - 1)^2} \overset{\frac{0-}{+\infty}}{=} 0-, \quad (\text{because } \lim_{x \to 0+} x^{\alpha} |\log x| \beta = 0),$$

$$f(e\pm) = \lim_{x \to e\pm} \frac{x \log x}{(\log x - 1)^2} \overset{\frac{e}{0+}}{=} +\infty, \implies x = e \text{ vertical asymptote},$$

$$f(+\infty) = \lim_{x \to +\infty} \frac{x \log x}{(\log x - 1)^2} \overset{\frac{\pm\infty}{+\infty}}{=} \lim_{x \to +\infty} \frac{x}{\log x} \frac{1}{\underbrace{\left(1 - \frac{1}{\log x}\right)^2}_{\longrightarrow 1}} = +\infty,$$

At $+\infty$ we could have an oblique asymptote $y = mx + q$. Being

$$m = \lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{\log x}{(\log x - 1)^2} = 0,$$

it follows that the asymptote does not exist.

*Continuity and differentiability:* $f$ is composition of functions continuous and differentiable in their domains, the same holds for $f$. We have

$$f'(x) = \left(\frac{x\log x}{(\log x-1)^2}\right)' = \frac{\left(\log x + x\frac{1}{x}\right)(\log x-1)^2 - x\log x\left(2(\log x-1)\frac{1}{x}\right)}{(\log x-1)^4}$$

$$= \frac{(\log x+1)(\log x-1) - 2\log x}{(\log x-1)^3} = \frac{(\log x)^2 - 2\log x - 1}{(\log x-1)^3}.$$

Being $f(0+) = 0$, we may extend $f$ by continuity from the right at $0$. Let us see if such an extension is also differentiable. To this aim, notice that

$$\lim_{x\to 0+} f'(x) = \lim_{x\to 0+} \frac{(\log x)^2 - 2\log x - 1}{(\log x-1)^3} \overset{\xi:=\log x}{=} \lim_{\xi\to-\infty} \frac{\xi^2 - 2\xi - 1}{(\xi-1)^3} = 0-,$$

so, continuous extension of $f$ at $0$ is also differentiable.

*Monotonicity, extreme points:* We have

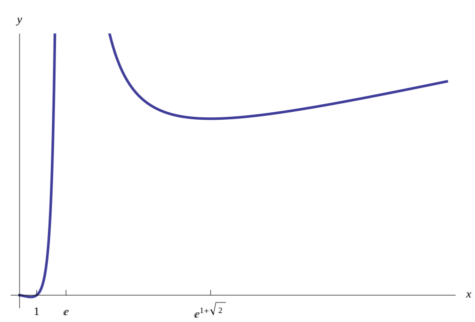
$$(\log x)^2 - 2\log x - 1 \geqslant 0, \iff \log x \leqslant 1-\sqrt{2}, \vee \log x \geqslant 1+\sqrt{2}, \iff x \leqslant e^{1-\sqrt{2}}, \vee x \geqslant e^{1+\sqrt{2}},$$

$$(\log x - 1)^3 \geqslant 0, \iff \log x - 1 \geqslant 0, \iff \log x \geqslant 1, \iff x \geqslant e.$$

By this we get the following table:

| | $0 \quad e^{1-\sqrt{2}}$ | $e^{1-\sqrt{2}} \quad e$ | $e \quad e^{1+\sqrt{2}}$ | $e^{1+\sqrt{2}} \quad +\infty$ |
|---|---|---|---|---|
| $\mathrm{sgn}(N)$ | $+$ | $-$ | $-$ | $+$ |
| $\mathrm{sgn}(D)$ | $-$ | $-$ | $+$ | $+$ |
| $\mathrm{sgn}(f')$ | $-$ | $+$ | $-$ | $+$ |
| $f$ | $\searrow$ | $\nearrow$ | $\searrow$ | $\nearrow$ |

Now: $f$ is continuous at $x = e^{1-\sqrt{2}}$ so this is a minimum point for $f$ on $]0, e[$. Similarly, $x = e^{1+\sqrt{2}}$ is a minimum for $f$ on $]e, +\infty[$. Moreover, because $f(e^{1-\sqrt{2}}) < 0 < f(e^{1+\sqrt{2}})$ we conclude that $x = e^{1-\sqrt{2}}$ is also a global minimum. There are not maximum points because $f$ is upper unbounded. □




Example 7.9.3. *Solve the inequality*

$$2x\log x + 1 \geqslant x^2.$$

Sol. — Consider the function $f(x) := 2x\log x + 1 - x^2$. Clearly, $D(f) = ]0, +\infty[$. Moreover

$$f(0+) = \lim_{x\to 0+}\left(2x\log x + 1 - x^2\right) = 1,$$

$$f(+\infty) = \lim_{x\to +\infty}\left(2x\log x + 1 - x^2\right) = \lim_{x\to +\infty} -x^2\left(1 - \frac{1}{x^2} - \frac{2}{x}\log x\right) = -\infty$$

being $\log x \ll_{+\infty} x$. Now, $f$ is continuous and differentiable on $]0, +\infty[$ and

$$f'(x) = 2\log x + 2x\frac{1}{x} - 2x = 2\log x + 2 - 2x = 2\left(\log x + 1 - x\right).$$

Therefore

$$f' \geqslant 0, \iff g(x) := \log x - x + 1 \geqslant 0.$$

This is a nontrivial inequality for which we may find two ways to answer. First way study the behaviour of function $g$ on $]0, +\infty[$. Namely, $g(0+) = -\infty$ and $g(+\infty) - \infty$. Moreover, $g$ is continuous and differentiable on $]0, +\infty[$ and

$$g'(x) = \frac{1}{x} - 1.$$

Therefore

$$g'(x) \geqslant 0, \iff \frac{1}{x} - 1 \geqslant 0, \iff \frac{1}{x} \geqslant 1, \overset{(x>0)}{\iff} x \leqslant 1.$$

Being $g$ continuous at $x = 1$ this is a global maximum for $g$. Being $g(1) = 0$ this means $g \leqslant 0$ for any $x \in ]0, +\infty[$.

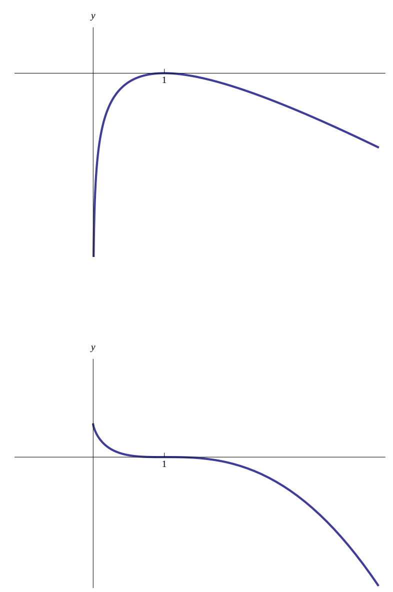
In the alternative, we may use the smart remark that log is concave. Therefore

$$\log x \leqslant \log 1 + (\log' 1)(x - 1) = 0 + (-1)(x - 1) = -x + 1,$$

for every $x \in ]0, +\infty[$. Therefore

$$\log x + x - 1 \leqslant 0, \ \forall x \in ]0, +\infty[.$$

In any case, we deduce $f' \leqslant 0$ on $]0, +\infty[$, thus in particular $f \searrow$. Finally, $f(1) = 0$, thus $f(x) > 0$ iff $x \in ]0, 1[$. □



Plots of $g$ (above) and $f$ (below).

## 7.10. Applications

Differential Calculus applies to solve many optimization problems. In this section, we explore some examples, but the number of applications is uncountable.

### 7.10.1. Constrained Optimization.

A typical applied problem consists in maximizing/minimizing a certain function where the variables are subject to some constraints.

Example 7.10.1. *Determine the radius $r$ and height $h$ of a cylindrical can having fixed volume $V$ in such a way that the surface $S$ is minimum.*

Sol. — The surface of the can is $S = 2\pi r^2 + 2\pi r h$, its volume is $V := \pi r^2 h$. Being $V$ fixed

$$h = \frac{V}{\pi r^2}, \implies S = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2} = 2\pi r^2 + \frac{2V}{r}.$$

Now

$$S'(r) = 4\pi r - \frac{2V}{r^2} = \frac{4\pi r^3 - 2V}{r^2} \geqslant 0, \iff 4\pi r^3 - 2V \geqslant 0, \iff r^3 \geqslant \frac{V}{2\pi}, \iff r \geqslant \sqrt[3]{\frac{V}{2\pi}}.$$

Therefore $S \searrow$ on $r \in ]0, \sqrt[3]{\frac{V}{2\pi}}]$, $S \nearrow$ on $[\sqrt[3]{\frac{V}{2\pi}}, +\infty[$. It follows that $r = \sqrt[3]{\frac{V}{2\pi}}$ is a minimum point for $S$. In such case

$$r = \sqrt[3]{\frac{V}{2\pi}}, \quad h = \frac{V}{\pi r^2} = \sqrt[3]{\frac{4V}{\pi}}.$$

A curiosity: if $V = 0,33\ dm^3 \equiv 333\ cm^3$ (as in the case of cans used for drinks), we found $r = 3,75\ cm$ and $h = 7,51\ cm$. These are sizes which minimize the cost of the can (and you can check by yourself how this goal is attained by drink producers. . . ). □

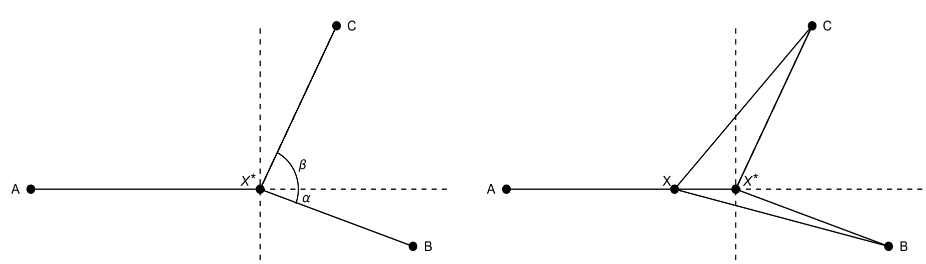
**7.10.2. A Logistic Problem.** An *Express Delivery Service* (such as FedEx, DHL, etc, but the problem is the same for on-line retailers such as Amazon, Alibaba, etc) has to place a logistics facility in such a way the delivering time (or fuel consumption) is minimized. The problem is of course where should be placed the facility. To simplify the solution to this complicated problem, we will assume that

- the facility serves just three towns, $A$, $B$ and $C$.
- A road is always available between any two points.
- The three cities have the same number of deliveries.

This last assumption, in particular, means that the problem consists in determining the position of a point $X$ such that

$$L(X) = \overline{AX} + \overline{BX} + \overline{CX},$$

be minimized. In certain cases, the situation is trivial, as when the three cities are aligned or if $ABC$ is obtuse. Let us see how to translate this problem into an optimization problem we may treat with the tools developed here. The first remark is the following: since distances are invariant by rotations and translations, we may always assume that the points $A$ and optimum $X^*$ belong to a Cartesian axis, for instance the $x-$axis.




Notice that $X^*$ must be in the interior of the triangle $ABC$, so in particular both $\alpha, \beta$ cannot be $> \frac{\pi}{2}$. Moreover, we may assume that the optimal $X^*$ corresponds to the origin. In other words,

$$L(X) \geqslant L(0,0),\ \forall X.$$

Let us now formalize a bit more in detail the situation. First, we introduce coordinates for points $A$, $B$ and $C$: $A = (a_1, 0)$, $B = (b_1, b_2)$ e $C = (c_1, c_2)$. We may always assume that $a_1 \leqslant b_1, c_1$, that is $A$ is at left of both $B$ and $C$. Second: consider $X$ on the $x-$axis, that is $X = (x, 0)$. Then

$$\begin{aligned} L(X) &= \sqrt{(a_1 - x)^2 + (0-0)^2} + \sqrt{(b_1 - x)^2 + b_2^2} + \sqrt{(c_1 - x)^2 + c_2^2} \\ &= |a_1 - x| + \sqrt{(b_1 - x)^2 + b_2^2} + \sqrt{(c_1 - x)^2 + c_2^2} =: L(x). \end{aligned}$$

Now, according to Fermat's theorem, $L'(0) = 0$. Notice that

$$L'(x) = \mathrm{sgn}(x - a_1) - \frac{b_1 - x}{\sqrt{(b_1 - x)^2 + b_2^2}} - \frac{c_1 - x}{\sqrt{(c_1 - x)^2 + c_2^2}}.$$

Therefore,

$$L'(0) = 0, \iff 1 - \frac{b_1}{\sqrt{b_1^2 + b_2^2}} - \frac{c_1}{\sqrt{c_1^2 + c_2^2}} = 0, \iff \frac{b_1}{\sqrt{b_1^2 + b_2^2}} + \frac{c_1}{\sqrt{c_1^2 + c_2^2}} = 1.$$

We notice that

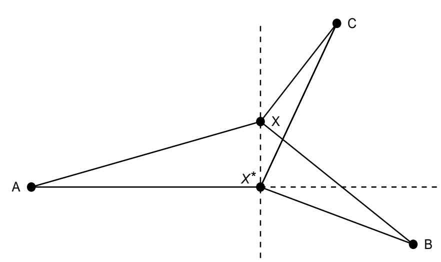
$$\frac{b_1}{\sqrt{b_1^2 + b_2^2}} = \cos\alpha, \quad \frac{c_1}{\sqrt{c_1^2 + c_2^2}} = \cos\beta.$$

Therefore, the previous relation reads

$$\boxed{\cos\alpha + \cos\beta = 1.}$$

Of course, this does not identify $\alpha$ and $\beta$, this because there are infinitely many couples $\alpha, \beta$ for which $\cos\alpha + \cos\beta = 1$.

Imagine now we take $X$ on the $y-$axis, $X = (0, y)$.




In this case, $X = (0, y)$ and

$$L(X) = \sqrt{a_1^2 + y^2} + \sqrt{b_1^2 + (b_2 - y)^2} + \sqrt{c_1^2 + (c_2 - y)^2} =: L(y).$$

Again, according to Fermat's theorem, $L'(0) = 0$. Since

$$L'(y) = \frac{y}{\sqrt{a_1^2 + y^2}} - \frac{b_2 - y}{\sqrt{b_1^2 + (b_2 - y)^2}} - \frac{c_2 - y}{\sqrt{c_1^2 + (c_2 - y)^2}},$$

we have

$$L'(0) = 0, \iff \frac{b_2}{\sqrt{b_1^2 + b_2^2}} + \frac{c_2}{\sqrt{c_1^2 + c_2^2}} = 0, \iff -\sin\alpha + \sin\beta = 0.$$

Therefore, $\alpha$ and $\beta$ must fulfil

$$\begin{cases} \cos\alpha + \cos\beta = 1, \\ \\ \sin\alpha = \sin\beta. \end{cases}$$

Recall now that $\alpha$ and $\beta$ are $\geqslant 0$ and cannot be both $> \frac{\pi}{2}$. Say that $0 \leqslant \alpha \leqslant \frac{\pi}{2}$. Then

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - \sin^2\beta},$$

and plugging this into the first equation

$$\sqrt{1 - \sin^2\beta} = 1 - \cos\beta, \iff 1 - \sin^2\beta = (1 - \cos\beta)^2 = 1 + \cos^2\beta - 2\cos\beta$$

that is

$$2\cos\beta = 1, \iff \cos\beta = \frac{1}{2}, \iff \beta = \frac{\pi}{3}.$$

But then, $\cos\alpha = \frac{1}{2}$ and $\sin\alpha = \frac{\sqrt{3}}{2}$, that is $\alpha = \frac{\pi}{3}$. We conclude that $X^*$ *is the point such that* $\widehat{AX^*B} = \widehat{AX^*C} = \widehat{BX^*C} = \frac{2\pi}{3}$. This point is called *Fermat point*, and this problem was addressed by Fermat as a geometrical problem in the XVII century. □

**7.10.3. Laws of Nature.** Physics is based on optimization principles. A beautiful example is the following

**Basic Axiom of Light Theory:** *light moves along paths minimizing the travel time.*

For example, in homogeneous media, light travels along straight lines. This principle has many consequences, one of which is the

**Theorem 7.10.2: Snellius' law of refraction**

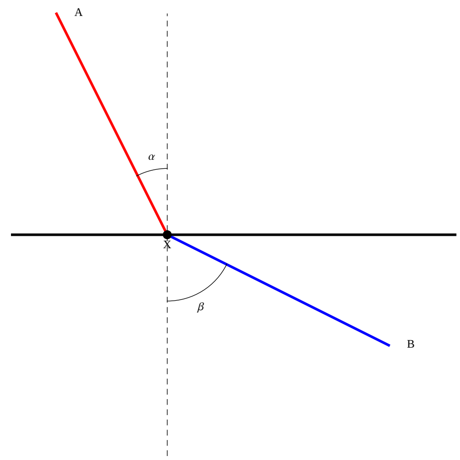
The ratio of the sines of the angle of incidence and refraction of a light ray traveling in two different homogeneous media is equal to the ratio of propagation speed in the respective media. In symbols

$$\frac{\sin\alpha}{\sin\beta} = \frac{v_1}{v_2}.$$

Proof. Consider a light ray travelling from the first media to the second one. In each media, rays are straight lines travelling at speed $v_1$ and $v_2$. Let fix point $A$ in the first media where the ray passes through) and a point $B$ in the second media where again the ray passes through. We represent the transition boundary from the first to the second media by a straight line that conventionally we may assume as an horizontal axis in a Cartesian plane. This assumption is not restrictive because no matter how is made the boundary between the two media, locally this looks like a straight line.

Therefore, denoting by $X$ the point where the ray crosses the boundary, the total travelling time from $A$ to $B$ is

$$\boxed{T = \frac{\overline{AX}}{v_1} + \frac{\overline{XB}}{v_2}.}$$




According to the basic axiom of Light Theory, $X$ must be such that $T$ is minimum. To translate this into a precise formulation, let $A = (a_1, a_2)$, $B = (b_1, b_2)$ and $X = (x, 0)$. We have

$$T = T(x) = \frac{\sqrt{(a_1 - x)^2 + a_2^2}}{v_1} + \frac{\sqrt{(x - b_1)^2 + b_2^2}}{v_2}.$$

It is easy to see that such $T$ has a unique minimum ($T(-\infty) = T(+\infty) = +\infty$, $T \in \mathscr{C}(\mathbb{R})$ assures that $T$ has a minimum). At a minimum point, according to Fermat's theorem, $T'(x) = 0$,

$$T'(x) = \frac{a_1 - x}{v_1\sqrt{(a_1 - x)^2 + a_2^2}} + \frac{x - b_1}{v_2\sqrt{(x - b_1)^2 + b_2^2}} = 0.$$

We may always assume that optimum is achieved at $x = 0$ (otherwise we translate everything),

$$0 = T'(0) = \frac{a_1}{v_1\sqrt{a_1^2 + a_2^2}} - \frac{b_1}{v_2\sqrt{b_1^2 + b_2^2}}.$$

On the other side

$$\frac{a_1}{\sqrt{a_1^2 + a_2^2}} = \sin\alpha, \quad \frac{b_1}{\sqrt{b_1^2 + b_2^2}} = \sin\beta,$$

so

$$T'(0) = 0, \iff \frac{\sin\alpha}{v_1} - \frac{\sin\beta}{v_2} = 0, \iff \frac{\sin\alpha}{\sin\beta} = \frac{v_1}{v_2}. \quad \square$$

**7.10.4. Newton's algorithm for the search of zeroes.** We return to an important applied problem we already discussed on continuity, that is, the solution of

$$f(x) = 0,$$

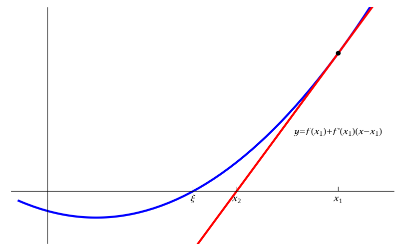
for some $f$. We already know that if $f \in \mathscr{C}(I)$, $I \subset \mathbb{R}$, is such that for some $a, b \in I$ we have $f(a)f(b) < 0$ (that is, $f$ takes opposite sign at $a$ and $b$) then a zero exists in the interval with endpoints

$a$ and $b$. We have seen a *bisection algorithm* to determine an approximated solution. A nice alternative is offered by Newton's algorithm, in the case when $f$ is *convex*.

To fix ideas, assume that $a < b$ and $f(a) < 0 < f(b)$, so we search for a zero in the interval $]a, b[$. Pick $x_1 = b$ and consider the tangent to $f$ at $(x_1, f(x_1))$, that is

$$y = f(x_1) + f'(x_1)(x - x_1).$$

First, notice that $f'(x_1) = f'(b) > 0$. Indeed, if $f'(b) \leqslant 0$ then, being $f$ convex, $f' \nearrow$ thus $f'(x) \leqslant 0$ for $x \in [a, b]$, that is $f \searrow$, thus $f(a) \geqslant f(b) > 0$, which is against our assumptions.




Since $f$ is convex, this tangent is below $f$ and it crosses the axis at point $x_2 > \xi$, where $\xi$ is the zero of $f$. This $x_2$ is given by

$$0 = f(x_1) + f'(x_1)(x_2 - x_1), \;\; x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} < x_1,$$

being $\frac{f(x_1)}{f'(x_1)} > 0$. We now repeat the argument starting by $x_2$: either $f(x_2) = 0$ or $f(x_2) > 0$. In the first case we stop, in the second we consider the tangent to $f$ at $(x_2, f(x_2))$, namely

$$y = f(x_2) + f'(x_2)(x - x_2).$$

Again, necessarily $f'(x_2) > 0$, otherwise $f$ would not have any zero on $[a, x_2]$ (but since $f(x_2) > 0 > f(a)$ that zero is there!), thus the tangent crosses the $x-$axis at

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} < x_2.$$

Iterating this procedure, we produce a sequence $x_1 > x_2 > x_3 > \ldots > x_n$ such that

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \;\; f'(x_n) \geqslant 0, \;\; f'(x_n) > 0.$$

Since $x_n \searrow$ there exists $\xi = \lim_n x_n$. Passing to the limit in the recurrence equation and assuming that $f \in \mathscr{C}^1(I)$ we have

$$\xi = \xi - \frac{f(\xi)}{f'(\xi)}, \quad \Longleftrightarrow \quad f(\xi) = 0.$$

## 7.11. Hôpital's rules

A typical problem in computing limits are indeterminate forms. Among these, there are $\frac{0}{0}$ and $\frac{\infty}{\infty}$ forms, that is to compute

$$\lim_{x \to x_0} \frac{f(x)}{g(x)},$$

where $f,g \longrightarrow 0$ or $f,g \longrightarrow \pm\infty$. Hôpital's rules are useful tools to treat this type of limits, transforming them into other and (hopefully!) easier limits.

**Theorem 7.11.1: $\frac{0}{0}$ rule**

Let $f,g : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, be defined in a neighbourhood $I_{x_0}\backslash\{x_0\}$ (that is defined around $x_0$, not necessarily at $x_0$) of $x_0 \in \mathbb{R}\cup\{\pm\infty\}$ and such that

i) $f,g$ are infinitesimal at $x_0$;
ii) $f,g$ are differentiable for $x \in I_{x_0}\backslash\{x_0\}$;
iii) $g'(x) \neq 0$ for $x \in I_{x_0}\backslash\{x_0\}$.

Then,

$$\exists \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = \ell \in \mathbb{R}\cup\{\pm\infty\}. \quad \Longrightarrow \quad \exists \lim_{x\to x_0} \frac{f(x)}{g(x)} = \ell.$$

Proof. We limit to the case $x_0 \in \mathbb{R}$. Let us do the proof for the limit as $x \longrightarrow x_0+$. Since $f,g$ are infinitesimal at $x_0$, we can extend both at $x_0$ by continuity (assigning to them the value 0). This way, we may write

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)}.$$

By Cauchy's theorem (our assumptions on $f,g$ assure that we can do it), there exists $\xi_x \in ]x_0,x[$ such that

$$\frac{f(x)-f(x_0)}{g(x)-g(x_0)} \overset{Cauchy\ 7.4}{=} \frac{f'(\xi_x)}{g'(\xi_x)}.$$

As $x \longrightarrow x_0+$, by two policemen thm, $\xi_x \longrightarrow x_0+$. Therefore, by our assumption,

$$\lim_{x\to x_0+} \frac{f(x)}{g(x)} = \lim_{x\to x_0+} \frac{f'(\xi_x)}{g'(\xi_x)} = \ell. \qquad \square$$

In common use, Hôpital's rules are applied according the following notation:

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} \overset{H}{=} \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

This is a conditioned equality:it holds true provided all hypotheses are verified.

Example 7.11.2. *Compute*

$$\lim_{x\to 0} \frac{\sinh x - x}{\sin(x^3)}.$$

Sol. — Let $f(x) := \sinh x - x$, $g(x) := \sin(x^3)$. Clearly, $f$ and $g$ are infinitesimal at 0, are differentiable on $\mathbb{R}$ and because

$$g'(x) = 3x^2\cos(x^3),$$

we have $g' = 0$ iff $x = 0$ or $x^3 = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. In particular, $g' \neq 0$in a $I_0\backslash\{0\}$, that is ii) holds. Moreover

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\cosh x - 1}{3x^2\cos(x^3)} = \lim_{x\to 0} \frac{1}{3\cos(x^3)} \frac{\cosh x - 1}{x^2} = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}.$$

Therefore, by first Hôpital's rule, the proposed limit exists and has value $\frac{1}{6}$. □

REMARK 7.11.3 (Warning!). A couple of important remarks:

- $\lim \frac{f}{g}$ **might exists while** $\lim \frac{f'}{g'}$ **may not!** For example,

$$\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} = \lim_{x\to 0} \frac{x}{\sin x} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x\to 0} 1_x \cdot x \sin \frac{1}{x} = 0$$

  by the bounded×infinitesimal rule. Applying the Hôpital's rule, however

$$\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} \overset{H}{=} \lim_{x\to 0} \frac{2x \sin \frac{1}{x} - \cos \frac{1}{x}}{\cos x} = -\lim_{x\to 0} \cos \frac{1}{x},$$

  that doesn't exist! This is not in contradiction with Hôpital's rule because this last provides just a **sufficient condition** in order that $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists.
- Hôpital's rules should never be applied in a blind mechanical way because they could lead to never-ending iterations. For instance, consider

$$\lim_{x\to 0+} \frac{e^{-1/x}}{x} \overset{\frac{0}{0},H}{=} \lim_{x\to 0+} \frac{\frac{1}{x^2} e^{-1/x}}{1} = \lim_{x\to 0+} \frac{e^{-1/x}}{x^2} \overset{\frac{0}{0},H}{=} \lim_{x\to 0+} \frac{\frac{1}{x^2} e^{-1/x}}{x^2} = \lim_{x\to 0+} \frac{e^{-1/x}}{x^4} = \ldots$$

  On the other hand

$$\lim_{x\to 0+} \frac{e^{-1/x}}{x} = \lim_{x\to 0+} \frac{\frac{1}{x}}{e^{1/x}} \overset{\frac{\infty}{\infty},H}{=} \lim_{x\to 0+} \frac{-\frac{1}{x^2}}{-\frac{1}{x^2} e^{1/x}} = \lim_{x\to 0+} \frac{1}{e^{1/x}} \overset{\frac{1}{+\infty}}{=} 0. \quad \square$$

Hôpital's rules may be iterated in their application.

EXAMPLE 7.11.4. *Compute*

$$\lim_{x\to 1} \left( \frac{1}{\log x} - \frac{1}{x-1} \right).$$

SOL. — Clearly

$$\lim_{x\to 1} \left( \frac{1}{\log x} - \frac{1}{x-1} \right) = \lim_{x\to 1} \frac{x - 1 - \log x}{(x-1)\log x}$$

where we recognize a form $\frac{0}{0}$. By the first Hôpital's rule,

$$\lim_{x\to 1} \frac{x - 1 - \log x}{(x-1)\log x} \overset{H}{=} \lim_{x\to 1} \frac{1 - \frac{1}{x}}{\log x + (x-1)\frac{1}{x}} = \lim_{x\to 1} \frac{x-1}{x \log x + x - 1} \overset{\frac{0}{0},H}{=} \lim_{x\to 1} \frac{1}{\log x + 1 + 1} = \lim_{x\to 1} \frac{1}{2 + \log x} = \frac{1}{2}. \quad \square$$

A similar rule holds for the $\frac{\infty}{\infty}$ form:

**Theorem 7.11.5: $\frac{\infty}{\infty}$ rule**

Let $f, g : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, be defined in a neighbourhood $I_{x_0} \backslash \{x_0\}$ (that is defined around $x_0$, not necessarily at $x_0$) of $x_0 \in \mathbb{R} \cup \{\pm\infty\}$ and such that

i) $f, g$ are infinite at $x_0$;
ii) $f, g$ are differentiable for $x \in I_{x_0} \backslash \{x_0\}$;
iii) $g'(x) \neq 0$ for $x \in I_{x_0} \backslash \{x_0\}$.

Then,

$$\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \ell \in \mathbb{R} \cup \{\pm\infty\}. \implies \exists \lim_{x \to x_0} \frac{f(x)}{g(x)} = \ell.$$

## 7.12. Taylor formula

Computing the fundamental limits we obtained formulas like

$$\sin x = x + o(x), \quad e^x = 1 + x + o(x).$$

These formulas say that sin and exp can be expressed as a first degree polynomial apart for an error which is smaller than $x$, therefore negligible respect to $x$ when $x$ is small. This principle is true in general and follows from the definition of differentiable function:

**Proposition 7.12.1**

Let $f$ be differentiable at $x_0$. Then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0). \tag{7.12.1}$$

Proof. Being differentiable

$$\exists f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \overset{x=x_0+h}{=} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

that is

$$0 = \lim_{x \to x_0} \left( \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right) = \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0}.$$

According to the definition of $o(\ldots)$,

$$f(x) - f(x_0) - f'(x_0)(x - x_0) = o(x - x_0),$$

which is the conclusion. □

This formula has an interesting numerical interpretation: *in the first approximation, any (differentiable) function $f$ is a first-order polynomial*. If we want to improve this formula, we naturally wonder if we can replace the first degree polynomial with a second, third, or generic $n$−th degree polynomial. We may expect that increasing the degree of the polynomial we should have a better approximation, that is, the error should be smaller. Let us see how we can address properly these questions.

A general $n-$th degree polynomial centred at $x_0$ has the form

$$\sum_{k=0}^{n} c_k (x - x_0)^k.$$

Thus, we search for coefficients $c_0, c_1, \ldots, c_n$ in such a way that

$$f(x) \approx_{x_0} \sum_{k=0}^{n} c_k (x - x_0)^k.$$

More precisely, we wish that the approximation error

$$\varepsilon(x) := f(x) - \sum_{k=0}^{n} c_k (x - x_0)^k$$

be small when $x \longleftarrow x_0$. Now, clearly all powers $(x - x_0)^k$ are small. Since $\varepsilon(x)$ should be negligible with respect to the polynomial, we guess that the proper form to say this should be

$$f(x) = \sum_{k=0}^{n} c_k (x - x_0)^k + o\left((x - x_0)^n\right). \tag{7.12.2}$$

Thus the question is: under which hypotheses (7.12.2) holds true? (notice that the case $n = 1$ is the (7.12.1)) And, in this case, how can we compute coefficients $c_k$?

It is easy to get a guess for this last question. Indeed: assume that $f$ itself is a polynomial and we look for $c_k$ in such a way that

$$f(x) = \sum_{k=0}^{n} c_k (x - x_0)^k.$$

We notice that $c_0 = f(x_0)$. Moreover, deriving

$$f'(x) = \sum_{k=1}^{n} k c_k (x - x_0)^{k-1}, \implies f'(x_0) = c_1.$$

To extract $c_2$ we derive $f'$: we get

$$f''(x) = \sum_{k=2} k(k-1) c_k (x - x_0)^{k-2}, \implies f''(x_0) = 2 \cdot 1 \cdot c_2, \iff c_2 = \frac{f''(x_0)}{2 \cdot 1} = \frac{f''(x_0)}{2!}.$$

Deriving again and calling $f''' = (f'')'$,

$$f'''(x) = \sum_{k=3} k(k-1)(k-2) c_k (x - x_0)^{k-3}, \implies f''(x_0) = 3 \cdot 2 \cdot 1 \cdot c_3, \iff c_3 = \frac{f'''(x_0)}{3!}.$$

We may now guess that $c_4 = \frac{f''''(x_0)}{4!}$ and so on. To proceed, we need to introduce a symbol: assuming this exists, for $k \geqslant 2$ we call $k-$**th derivative of** $f$ **at** $x_0$ (or **derivative of order** $k$)

$$f^{(k)}(x_0) = (f^{(k-1)})'(x_0).$$

For convenience, we set also

$$f^{(0)} \equiv f.$$

With this notation, the above argument shows that

$$\text{if } f(x) = \sum_{k=0}^{n} c_k (x-x_0)^k, \implies f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

This is the right polynomial in general:

**Theorem 7.12.2: Peano**

Let $f$ be differentiable $n-1$ times in $I_{x_0}$ and such that exists $f^{(n)}(x_0)$. Then

(7.12.3) $$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + o((x-x_0)^n). \quad \text{(Taylor formula)}$$

The polynomial is called **Taylor polynomial of order $n$ centred at $x_0$**.

Proof. We will limit to the case $n=2$. We have to prove that

$$\lim_{x\to x_0} \frac{f(x) - \left(f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2\right)}{(x-x_0)^2} = 0.$$

This is nothing but a form $\frac{0}{0}$. Applying the Hôpital's rule we have

$$\lim_{x\to x_0} \frac{f(x) - \left(f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2\right)}{(x-x_0)^2} \overset{H}{=} \lim_{x\to x_0} \frac{f'(x) - (f'(x_0) + f''(x_0)(x-x_0))}{2(x-x_0)}$$

$$= \frac{1}{2}\lim_{x\to x_0}\left(\frac{f'(x)-f'(x_0)}{x-x_0} - f''(x_0)\right) = 0,$$

because of the definition of $f''(x_0) = (f')'(x_0)$. □

A remarkable case of Taylor formula is the **McLaurin formula**, which is nothing but the Taylor formula centred at $x_0 = 0$:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + o(x^n).$$

Let's see how this formula works in the case of

Example 7.12.3 (exponential).

(7.12.4) $$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n), \ \forall n \in \mathbb{N}.$$

SOL. — Let $f(x) := e^x$. Clearly, $f$ is derivable infinitely many times being $f' = f$. This means that the McLaurin formula can be written up to any order $n$. We have

| $k$ | $f^{(k)}(x)$ | $f^{(k)}(0)$ |
|---|---|---|
| 0 | $e^x$ | 1 |
| 1 | $(e^x)' = e^x$ | 1 |
| 2 | $(e^x)' = e^x$ | 1 |
| 3 | $(e^x)' = e^x$ | 1 |
| $\vdots$ | $\vdots$ | $\vdots$ |

Therefore $f^{(k)}(0) = 1$ for every $k$, hence the (7.12.4) follows. □

EXAMPLE 7.12.4 (sinh, cosh).

$$\sinh x = \sum_{j=0}^{m} \frac{x^{2j+1}}{(2j+1)!} + o(x^{2m+1}), \quad \cosh x = \sum_{j=0}^{m} \frac{x^{2j}}{(2j)!} + o(x^{2m}), \quad \forall m \in \mathbb{N} \tag{7.12.5}$$

SOL. — Let $f = \sinh$, $g = \cosh$. Being $f' = g$, $g' = f$ we have that $f$ and $g$ are derivable infinitely many times, hence the McLaurin expansions holds up to any order $n$. We have

| $k$ | $f^{(k)}(x)$ | $f^{(k)}(0)$ |
|---|---|---|
| 0 | $\sinh x$ | 0 |
| 1 | $(\sinh x)' = \cosh x$ | 1 |
| 2 | $(\cosh x)' = \sinh x$ | 0 |
| 3 | $(\sinh x)' = \cosh x$ | 1 |
| $\vdots$ | $\vdots$ | $\vdots$ |

| $k$ | $g^{(k)}(x)$ | $g^{(k)}(0)$ |
|---|---|---|
| 0 | $\cosh x$ | 1 |
| 1 | $(\cosh x)' = \sinh x$ | 0 |
| 2 | $(\sinh x)' = \cosh x$ | 1 |
| 3 | $(\cosh x)' = \sinh x$ | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ |

We see that $f^{(k)}(0) = 0, 1$ according to $k$ even or odd. From this follows that the McLaurin expansion contains only powers with odd exponent. By choosing $n = 2m + 1$ we get easily the first of the (7.12.5). Similarly for $g$. □

EXAMPLE 7.12.5 (sin, cos).

$$\sin x = \sum_{j=0}^{m} (-1)^j \frac{x^{2j+1}}{(2j+1)!} + o(x^{2m+1}), \quad \cos x = \sum_{j=0}^{m} (-1)^j \frac{x^{2j}}{(2j)!} + o(x^{2m}), \quad \forall m \in \mathbb{N}. \tag{7.12.6}$$

SOL. — Let $f = \sin$, $g = \cos$. Here, $f$ and $g$ are both derivable infinitely many times being $f' = (\sin x)' = \cos x = g$ and $g' = (\cos x)' = -\sin x = -f$. To compute the coefficients, notice that

| $k$ | $f^{(k)}(x)$ | $f^{(k)}(0)$ |
|---|---|---|
| 0 | $\sin x$ | 0 |
| 1 | $(\sin x)' = \cos x$ | 1 |
| 2 | $(\cos x)' = -\sin x$ | 0 |
| 3 | $(-\sin x)' = -\cos x$ | $-1$ |
| 4 | $(-\cos x)' = \sin x$ | 0 |
| 5 | $(\sin x)' = \cos x$ | 1 |
| $\vdots$ | $\vdots$ | $\vdots$ |

| $k$ | $g^{(k)}(x)$ | $g^{(k)}(0)$ |
|---|---|---|
| 0 | $\cos x$ | 1 |
| 1 | $(\cos x)' = -\sin x$ | 0 |
| 2 | $(-\sin x)' = -\cos x$ | $-1$ |
| 3 | $(-\cos x)' = \sin x$ | 0 |
| 4 | $(\sin x)' = \cos x$ | 1 |
| 5 | $(\cos x)' = -\sin x$ | $-1$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

Therefore $f^{(k)}(0) = 0$ for $k$ even, $f^{(k)}(0) = \pm 1$ for $k$ odd. Precisely, we see that writing $k = 2j + 1$ we have $f^{(2j+1)}(0) = (-1)^j$. From this, the first of the (7.12.6) follows and similarly the second is obtained. □

EXAMPLE 7.12.6 (logarithm).

$$\log(1+x) = \sum_{k=1}^{n} (-1)^{k-1}\frac{x^k}{k} + o(x^n). \tag{7.12.7}$$

SOL. — Let $f(x) = \log(1+x)$. Clearly, $f$ is defined and derivable for $x = 0$. By looking at the derivatives of $f$ it is easy to check that it admits derivatives of any order. To compute the coefficients, notice that

| $k$ | $f^{(k)}(x)$ | $f^{(k)}(0)$ |
|---|---|---|
| 0 | $\log(1+x)$ | 0 |
| 1 | $(\log(1+x))' = \frac{1}{1+x} = (1+x)^{-1}$ | 1 |
| 2 | $((1+x)^{-1})' = -(1+x)^{-2}$ | $-1$ |
| 3 | $(-(1+x)^{-2})' = 2(1+x)^{-3}$ | 2 |
| 4 | $(2(1+x)^{-3})' = -3\cdot 2(1+x)^{-4}$ | $-3\cdot 2$ |
| 5 | $(-3\cdot 2(1+x)^{-4})' = 4\cdot 3\cdot 2(1+x)^{-4}$ | $4\cdot 3\cdot 2$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

Apart for the first coefficient (null) the others have alternate sign $+,-,+,-,\dots$ and absolute value $1, 1, 2, 3\cdot 2, 4\cdot 3\cdot 2, \dots$. Precisely: $f^{(k)}(0) = (-1)^{k-1}(k-1)!$ hence

$$\log(1+x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{k!}x^k + o(x^n) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}x^k + o(x^n). \qquad \square$$

REMARK 7.12.7 (Warning!). **By McLaurin formula for** log **we do not mean the McLaurin formula for** $f(x) = \log x$ **which is meaningless** ($\log x$ as well as its derivatives are not even defined at $x = 0$). We mean the McLaurin formula for $\log(1+x)$. In short, McLaurin formula for $\log(1+x)$ corresponds to the Taylor formula centred at $x = 1$ for $\log x$. $\square$

EXAMPLE 7.12.8 (power).

(7.12.8)

$$(1+x)^\alpha = 1 + \sum_{k=1}^{n} \begin{pmatrix} \alpha \\ k \end{pmatrix} x^k + o(x^n), \ \ dove \begin{pmatrix} \alpha \\ k \end{pmatrix} := \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}, \ \forall\alpha\in\mathbb{R},\ \forall n\in\mathbb{N}.$$

SOL. — Let $f(x) = (1+x)^\alpha$. We have

| $k$ | $f^{(k)}(x)$ | $f^{(k)}(0)$ |
|---|---|---|
| 0 | $(1+x)^\alpha$ | 1 |
| 1 | $((1+x)^\alpha)' = \alpha(1+x)^{\alpha-1}$ | $\alpha$ |
| 2 | $(\alpha(1+x)^{\alpha-1})' = \alpha(\alpha-1)(1+x)^{\alpha-2}$ | $\alpha(\alpha-1)$ |
| 3 | $\left(\alpha(\alpha-1)(1+x)^{\alpha-2}\right)' = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$ | $\alpha(\alpha-1)(\alpha-2)$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

Now the conclusion follows easily. $\square$

REMARK 7.12.9 (Warning!). **As for logarithm, McLaurin expansion of the power is not McLaurin formula for** $x^\alpha$ **but for** $(1+x)^\alpha$. $\square$

Example 7.12.10. *Compute McLaurin asymptotic expansion to the fourth order of the function*

$$f(x) = \log(1+e^x)$$

Sol. — We need the first four derivatives of $f$:

$$\begin{aligned}
f'(x) &= \frac{e^x}{1+e^x},\\
f''(x) &= \frac{e^x(1+e^x)-e^xe^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2},\\
f'''(x) &= \frac{e^x(1+e^x)^2-e^x2(1+e^x)e^x}{(1+e^x)^4} = e^x\frac{1+e^x-2e^x}{(1+e^x)^3} = \frac{e^x-e^{2x}}{(1+e^x)^3},\\
f''''(x) &= \frac{(e^x-2e^{2x})(1+e^x)^3-(e^x-e^{2x})3(1+e^x)^2e^x}{(1+e^x)^6} = e^x\frac{(1-2e^x)(1+e^x)-3(1-e^x)}{(1+e^x)^4}
\end{aligned}$$

By this

$$f(0)=\log 2,\ \ f'(0)=\frac{1}{2},\ \ f''(0)=\frac{1}{4},\ \ f'''(0)=0,\ \ f''''(0)=-\frac{1}{8}.$$

Therefore

$$\log(1+e^x) = \log 2+\frac{1}{2}x+\frac{1/4}{2!}x^2+\frac{0}{3!}x^3+\frac{-1/8}{4!}x^4+o(x^4) = \frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{224}x^4+o(x^4). \quad \square$$

**7.12.1. Computing limits by using McLaurin formulas.** In this subsection, we present a new powerful method in computing limits. We will introduce it by a first example:

Example 7.12.11. *Compute*

$$\lim_{x\to 0}\frac{(1-\cos(3x))^2}{x^2(1-\cos x)}.$$

Sol. — Call $N$ and $D$ numerator and denominator. Because $3x \longrightarrow 0$ and $\cos t = 1-\frac{t^2}{2}+o(t^2)$ as $t \longrightarrow 0$ (we choose the "shortest" not trivial expansion to write the minor number of terms), we have

$$N(x) = (1-\cos(3x))^2 = \left(1-\left(1-\frac{(3x)^2}{2}+o\left((3x)^2\right)\right)\right)^2 = \left(\frac{9}{2}x^2+o(9x^2)\right)^2 = \frac{81}{4}x^4+9x^2o(9x^2)+o(9x^2)^2.$$

Now, let us look at terms $9x^2o(x^2)$ and $o(9x^2)^2$. The first is $x^2$ times something smaller than $x^2$. It sounds reasonable that

$$9x^2o(9x^2) = o(x^4).$$

Is that true? Recall that $f=o(g)$ means $f/g \longrightarrow 0$. Thus, we have to check if

$$\frac{9x^2o(9x^2)}{x^4} \longrightarrow 0.\ \ \text{But:}\ \frac{9x^2o(9x^2)}{x^4} = 9\frac{o(9x^2)}{x^2} = 81\frac{o(9x^2)}{9x^2} \longrightarrow 0,$$

because $\frac{o(\heartsuit)}{\heartsuit} \longrightarrow 0$ if $\heartsuit \longrightarrow 0$. By the same intuition, it seems reasonable that

$$o(9x^2)^2 = o(x^4),\ \ \text{Indeed,}\ \frac{o(9x^2)^2}{x^2} = \frac{o(9x^2)}{x^2}\frac{o(9x^2)}{x^2} = 81\frac{o(9x^2)}{9x^2}\frac{o(9x^2)}{9x^2} \longrightarrow 0\cdot 0 = 0.$$

We conclude that

$$N(x) = \frac{81}{4}x^4 + o(x^4) + o(x^4) = \frac{81}{4}x^4 + o(x^4),$$

because it is evident that $o(x^4) + o(x^4) = o(x^4)$. We can now say that $N(x) \sim \frac{81}{4}x^4$ because

$$N(x) = \frac{81}{4}x^4\left(1 + \frac{4}{81}\frac{o(x^4)}{x^4}\right) = \frac{81}{4}x^4 \cdot 1_x.$$

In other words: we reduced the numerator to some power! If we can do the same for the denominator, we are done because it is much easier to compare powers than complicate expressions.

$$D(x) = x^2(1-\cos x) = x^2\left(1 - \left(1 - \frac{x^2}{2} + o(x^2)\right)\right) = x^2\left(\frac{x^2}{2} + o(x^2)\right) = \frac{1}{2}x^4 + x^2 o(x^2) = \frac{1}{2}x^4 + o(x^4) = \frac{x^4}{2}\cdot 1_x.$$

Therefore

$$\frac{N(x)}{D(x)} = \frac{\frac{81}{4}x^4 \cdot 1_x}{\frac{x^4}{2}\cdot 1_x} = \frac{81}{2}\cdot 1_x \longrightarrow 1. \quad \square$$

In the previous example, we met some of the *rules of calculus with infinitesimal quantities*. They are easy to understand and the reader is invited to develop some intuitive numerical sense on them.

**Proposition 7.12.12**

As $x \longrightarrow 0$:

- $o(x) + o(x) = o(x)$;
- $co(x) = o(cx) = o(x)$ for any $c \in \mathbb{R}\backslash\{0\}$;
- $x^n = o(x^m)$ if $n > m$ (also $n, m$ reals, in this case $x \longrightarrow 0+$ to make sense);
- $o(x^n) = o(x^m)$ if $n \geqslant m$ (also $n, m$ reals, in this case $x \longrightarrow 0+$ to make sense);
- $x^n o(x^m) = o(x^{n+m})$;
- $o(x^n)o(x^m) = o(x^{n+m})$;
- $(x + o(x))^n = x^n + o(x^n)$ (also $n$ real, in this case $x \longrightarrow 0+$ to make sense);
- $o(x + o(x)) = o(x)$.

Proof. i) and ii) are easy (exercise). iii): $x^n = o(x^m)$ as $n > m$ iff $\frac{x^n}{x^m} \longrightarrow 0$. But

$$\frac{x^n}{x^m} = x^{n-m} \longrightarrow 0, \; (n > m).$$

iv), v) and vi) are similar. vii) We will limit to the case $n \in \mathbb{N}$. By Newton binomial formula

$$(x + o(x))^n = \sum_{k=0}^{n}\binom{n}{k}x^k o(x^{n-k}) = x^n + \sum_{k=0}^{n-1}\binom{n}{k}o(x^n) = x^n + o(x^n),$$

by previous properties.

vii) is more delicate. We have to prove that $\frac{o(x+o(x))}{x} \longrightarrow 0$. It would be natural to write

$$\frac{o(x+o(x))}{x} = \frac{o(x+o(x))}{x+o(x)}\frac{x+o(x)}{x} \longrightarrow 0,$$

but we have to be careful with division by 0. However, this is easily solved noticing that

$$x + o(x) = x\left(1 + \frac{o(x)}{x}\right) = x \cdot 1_x,$$

and because $1_x \longrightarrow 1$, $1_x \neq 0$ in some $I_0 \backslash \{0\}$: but then $x + o(x) = x \cdot 1_x \neq 0$ as $x \in I_0 \backslash \{0\}$, and this authorizes previous passages. □

Remark 7.12.13 (Warning!). *All these properties have the form $\diamond = o(\heartsuit)$. This means $\frac{\diamond}{\heartsuit} \longrightarrow 0$ as $\heartsuit \longrightarrow 0$. The order is important and it cannot be inverted!* For instance, $o(x^2) = o(x)$ is true but of course $o(x) = o(x^2)$ is false! □

Example 7.12.14. *Compute*

$$\lim_{x \to 0+} \frac{\sqrt[4]{\cos x} - e^{-x^2}}{\sqrt{x}\log(1 + x\sin\sqrt{x}) - x^3 + x(e^x - 1)}.$$

Sol. — Immediately we recognize a form $\frac{0}{0}$. Recalling that

$$\cos t = 1 - \frac{t^2}{2} + o(t^2), \quad e^t = 1 + t + o(t), \quad (1+t)^\alpha = 1 + \alpha t + o(t),$$

we have

$$\begin{aligned} N(x) &= \sqrt[4]{\cos x} - e^{-x^2} = \left(1 - \frac{x^2}{2} + o(x^2)\right)^{1/2} - \left(1 - \frac{x^2}{2} + o(x^2)\right) \\ &= \left(1 + \frac{1}{2}\left(-\frac{x^2}{2} + o(x^2)\right) + o\left(-\frac{x^2}{2} + o(x^2)\right)\right) - 1 + \frac{x^2}{2} + o(x^2). \\ &= \frac{x^2}{4} + o(x^2) + o\left(-\frac{x^2}{2} + o(x^2)\right). \end{aligned}$$

By rules of calculus, $o\left(-\frac{x^2}{2} + o(x^2)\right) = o(x^2)$, so $N(x) = \frac{x^2}{2} + o(x^2) = \frac{x^2}{2} \cdot 1_x$.

Passing to the denominator,

$$\sin t = t + o(t), \quad \log(1+t) = t + o(t),$$

therefore

$$\begin{aligned} D(x) &= \sqrt{x}\log\left(1 + x\left(\sqrt{x} + o(\sqrt{x})\right)\right) - x^3 + x\left(1 + x + o(x) - 1\right) = x^{1/2}\log\left(1 + x^{3/2} + o(x^{3/2})\right) - x^3 + x^2 + o(x^2) \\ &= x^{1/2}\left(x^{3/2} + o(x^{3/2}) + o\left(x^{3/2} + o(x^{3/2})\right)\right) - x^3 + x^2 + o(x^2) \\ &= x^2 + o(x^2) - x^3 + x^2 = 2x^2 + o(x^2) = 2x^2 \cdot 1_x, \end{aligned}$$

being $x^3 = o(x^2)$. In conclusion

$$\frac{N(x)}{D(x)} = \frac{\frac{x^2}{2} \cdot 1_x}{2x^2 \cdot 1_x} = \frac{1}{4} \cdot 1_x \longrightarrow 1_x. \quad □$$

Example 7.12.15. *Compute, for $\alpha > 0$, the limit*

$$\lim_{x\to 0^+} \frac{\log\left(1+x^\alpha\right) - \sin(x^8)}{e^{x^{2\alpha}} - 1}$$

Sol. — Being $\alpha > 0$, $x^\alpha \longrightarrow 0+$ as $x \longrightarrow 0+$, so we recognize a form $\frac{0}{0}$. Recalling that

$$\left.\begin{aligned} \log(1+\xi) &= \xi - \frac{\xi^2}{2} + \ldots + (-1)^{n+1}\frac{\xi^n}{n} + o(\xi^n), \\ \sin\xi &= \xi - \frac{\xi^3}{3!} + \ldots + (-1)^n \frac{\xi^{2n+1}}{(2n+1)!} + o(\xi^{2n+1}), \\ e^\xi &= \xi + \frac{\xi^2}{2!} + \ldots + \frac{\xi^n}{n!} + o(\xi^n), \end{aligned}\right\} \text{ as } \xi \to 0.$$

we have

$$N(x) = x^\alpha + o(x^\alpha) - (x^8 + o(x^8)) = x^\alpha - x^8 + o(x^\alpha) + o(x^8).$$

There're three cases: $\alpha < 8$, $\alpha = 8$ and $\alpha > 8$. In the first one

$$x^8 + o(x^8) = o(x^\alpha), \implies N(x) = x^\alpha + o(x^\alpha).$$

If $\alpha = 8$ we have $N(x) = o(x^8)$ which is too vague to know the precise behavior of the numerator. To solve the impasse, we extend the asymptotic expansion for log and sin to get

$$N(x) = x^8 - \frac{(x^8)^2}{2} + o((x^8)^2) - \left(x^8 - \frac{(x^8)^3}{3!} + o((x^8)^3)\right) = -\frac{x^{16}}{2} + o(x^{16}).$$

Finally, as $\alpha > 8$, we have $x^\alpha + o(x^\alpha) = 0(x^8)$, so

$$N(x) = x^8 + o(x^8).$$

Summarizing:

$$N(x) = \begin{cases} x^\alpha \cdot 1_x, & \alpha < 8, \\ -\frac{1}{2}x^{16}\cdot 1_x, & \alpha = 8, \\ x^8 \cdot 1_x, & \alpha > 8. \end{cases}$$

About the denominator the discussion is easier:

$$D(x) = x^{2\alpha} + o(x^{2\alpha}) = x^{2\alpha}\cdot 1_x.$$

Therefore

$$\frac{N(x)}{D(x)} = \begin{cases} \frac{x^\alpha \cdot 1_x}{x^{2\alpha}\cdot 1_x} = x^{-\alpha}\cdot 1_x \longrightarrow +\infty, & \alpha < 8, \\ = \frac{-\frac{1}{2}x^{16}\cdot 1_x}{x^{16}\cdot 1_x} = -\frac{1}{2}\cdot 1_x \longrightarrow -\frac{1}{2}, & \alpha = 8, \\ = \frac{x^8\cdot 1_x}{x^{2\alpha}\cdot 1_x} = x^{8-2\alpha}\cdot 1_x \longrightarrow +\infty, & \alpha > 8. \end{cases} \qquad \square$$

**7.12.2. Applications to convergence of numerical series.** The method we introduced in the previous subsection is quite flexible and can be used fruitfully for the convergence of series. Let us see this for an example.

Example 7.12.16. *Determine $\alpha > 0$ such that the series*

$$\sum_{n=1}^{\infty} n^2 \left(1 - \cos\frac{1}{n}\right)\left(\frac{1}{n^\alpha} - \sin\frac{1}{n}\right),$$

*converges.*

Sol. — Notice that the two parentheses are infinitesimal as $n \longrightarrow +\infty$. The question is: how much are they small? Let us use the asymptotic expansion to answer: recall first that

$$\cos x = 1 - \frac{x^2}{2} + o(x^2), \quad \sin x = x + o(x) = x - \frac{x^3}{6} + o(x^3), \quad x \to 0,$$

so

$$1 - \cos\frac{1}{n} = \frac{(1/n)^2}{2} + o\left(\left(\frac{1}{n}\right)^2\right) = \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) \sim \frac{1}{2n^2},$$

while

$$\frac{1}{n^\alpha} - \sin\frac{1}{n} = \frac{1}{n^\alpha} - \left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right) = \frac{1}{n^\alpha} - \frac{1}{n} + o\left(\frac{1}{n}\right).$$

Here we have three cases: $0 < \alpha < 1$, $\alpha = 1$ and $\alpha > 1$. If $0 < \alpha < 1$ then

$$\frac{1}{n} = o\left(\frac{1}{n^\alpha}\right), \quad o\left(\frac{1}{n}\right) = o\left(\frac{1}{n^\alpha}\right), \implies \frac{1}{n^\alpha} - \sin\frac{1}{n} = \frac{1}{n^\alpha} + o\left(\frac{1}{n^\alpha}\right) \sim \frac{1}{n^\alpha}.$$

Therefore, if $0\alpha < 1$ we have

$$a_n = n^2 \frac{1}{2n^2}\frac{1}{n^\alpha}1_n \sim \frac{1}{2n^\alpha}.$$

In the case $\alpha = 1$ the expansion for sin is not enough because we have

$$\frac{1}{n} - \sin\frac{1}{n} = o\left(\frac{1}{n}\right),$$

which is almost useless. Extending the expansion for sin we have

$$\frac{1}{n} - \sin\frac{1}{n} = \frac{1}{n} - \left(\frac{1}{n} - \frac{(1/n)^3}{6} + o\left(\left(\frac{1}{n}\right)^3\right)\right) = \frac{1}{6n^3} + o\left(\frac{1}{n^3}\right) \sim \frac{1}{6n^3},$$

so

$$a_n = n^2 \frac{1}{2n^2}\frac{1}{6n^3}1_n \sim \frac{1}{12n^3}.$$

Finally, if $\alpha > 1$, we have

$$\frac{1}{n^\alpha} = o\left(\frac{1}{n}\right), \implies \frac{1}{n^\alpha} - \sin\frac{1}{n} = -\frac{1}{n} + o\left(\frac{1}{n}\right) \sim -\frac{1}{n},$$

hence

$$a_n = n^2 \frac{1}{2n^2}\left(-\frac{1}{n}\right)1_n \sim -\frac{1}{2n}.$$

Summarizing:

$$a_n \sim \begin{cases} \frac{1/2}{n^\alpha}, & 0 < \alpha < 1 \quad \text{the series converges iff } \alpha > 1 \text{ by asymptotic comparison, hence never in this case;} \\ \\ \frac{1/12}{n^3}, & \alpha = 1 \quad \text{the series converges by asymptotic comparison;} \\ \\ -\frac{1}{n}, & \alpha > 1 \quad \text{the series diverges by asymptotic comparison.} \end{cases}$$

□

## 7.13. Taylor Series

According to Taylor formula, if $f$ is $n-$times derivable at $x_0$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + o\left((x - x_0)^n\right).$$

In many cases, $n$ can be taken whatever natural. Since $f^{(n)} = (f^{(n-1)})'$, automatically $f^{(n-1)} \in \mathscr{C}$ just because it is derivable.

**Definition 7.13.1**

We say that $f \in \mathscr{C}^\infty(D)$ if there exists $f^{(n)}$ on $D$ for every $n \in \mathbb{N}$. In this case, $f^{(n)} \in \mathscr{C}(D)$ for every $n \in \mathbb{N}$.

Most of the elementary functions are $\mathscr{C}^\infty$ functions. Thus, for example: $e^x, \sin x, \cos x, \sinh x,\ \cosh x \in \mathscr{C}^\infty(\mathbb{R})$. In addition, $\log(1+x), (1+x)^\alpha \in \mathscr{C}^\infty(]-1, +\infty[$.

Bringing Taylor formula to extreme consequences, one may wonder if

$$f \in \mathscr{C}^\infty(I_{x_0}), \implies f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

This type of series is called **Taylor series**. In general, however, this is false.

Example 7.13.2. *Let*

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ \\ 0, & x = 0. \end{cases}$$

*Then $f \in \mathscr{C}^\infty(\mathbb{R})$ but its Taylor series vanishes. In particular, $f$ does not coincide with it.*

Sol. — We limit to sketch the answer, the details might be a bit tedious. First, there is no doubt that $f \in \mathscr{C}^\infty(\mathbb{R}\backslash\{0\})$. Easily, $f$ is continuous at 0. To check $f$ differentiable at 0, we compute

$$f'(x) \overset{x \neq 0}{=} e^{-1/x^2} \frac{2}{x^3} \overset{x \longrightarrow 0}{\longrightarrow} 0$$

so $\exists f'(0) = 0$ according to the differentiability test. Automatically, $f'$ is continuous at 0, thus $f' \in \mathscr{C}(\mathbb{R})$. Iterating this argument, we can show that all derivatives $f^{(n)}$ exist and are continuous on $\mathbb{R}$ and $f^{(n)}(0) = 0$ for every $n$. Thus, in particular

$$f(x) \neq \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \equiv 0. \quad \square$$

Despite this, it seems reasonable that, under suitable assumptions, a $\mathscr{C}^\infty$ coincides with the sum of its Taylor series:

**Theorem 7.13.3**

Let $f \in \mathscr{C}^\infty(I_{x_0})$ be such that

$$\exists M > 0, : \ |f^{(n)}(x)| \leqslant M^n, \ \forall x \in I_{x_0}.$$

Then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \ \forall x \in I_{x_0}.$$

We omit the proof here. It is easy to check that the principal elementary functions verify the previous theorem assumptions. In particular, we have remarkable formulas:

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, $\quad \sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$, $\quad \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$

$\log(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$, $\quad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$, $\quad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

## 7.14. Exercises

Exercise 7.14.1. Let

$$f(x) := \begin{cases} x \sin \dfrac{1}{x}, & x \in \mathbb{R}\backslash\{0\}, \\ 0, & x = 0. \end{cases}$$

Show that i) $f$ is continuous at $x = 0$; ii) is not differentiable at $x = 0$.

Exercise 7.14.2. Let

$$f(x) := \begin{cases} x^2 \sin \dfrac{1}{x}, & x \in \mathbb{R}\backslash\{0\}, \\ 0, & x = 0. \end{cases}$$

i) Show that $f$ is continuous at $x = 0$. ii) Is it differentiable at $x = 0$?

Exercise 7.14.3. Let

$$f(x) := \begin{cases} \sin x, & -1 \leqslant x < 0, \\ \dfrac{(\sin x^2)^5}{(\tan x^3)^2}, & 0 < x \leqslant 1. \end{cases}$$

Say if $f$ is extendable continuously at $x = 0$. In this case, is the extension differentiable at $x = 0$?

EXERCISE 7.14.4. Using carefully the rules of calculus, say where the following functions are differentiable:

1. $\dfrac{x^2-2x-1}{x-1}$. 2. $\sin(\log x)$. 3. $(1+e^x)^3$. 4. $(\sin x)^{\cos x}$. 5. $\log\left(1+|\log|x||\right)$.

6. $\sqrt{|x|^3+1}$. 7. $|x+1|^{\frac{x}{x-1}}$. 8. $\sqrt{\sin\sqrt{x}}$. 9. $\cosh|x|$. 10. $\sinh|x|$.

EXERCISE 7.14.5. For any of the following functions, say if they are differentiable at $x=0$:

1. $e^{-|x|}$. 2. $x|x|$. 3. $|x\sin x|$. 4. $x[x-1]$. 5. $[x](x-1)$.

EXERCISE 7.14.6. Compute the derivatives:

1. $x^6-2x^3+6x$. 2. $\frac{4}{x^3}+5x^4-\frac{7}{x^5}+\frac{1}{x^8}$. 3. $\left(x^3-\frac{1}{x^3}+3\right)^4$. 4. $\left(\frac{1+x^2}{1+x}\right)^5$.
5. $\sqrt{x^2+1}-x$. 6. $\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}$. 7. $\frac{\sqrt{x}}{\sqrt[3]{1+\sqrt{x}}}$. 8. $\sqrt[4]{x\sqrt[3]{x\sqrt{x}}}$.

9. $(\sin x)^3-3\sin x$. 10. $\frac{x^2}{\cos x}$. 11. $\tan x+\frac{1}{\cos x}$. 12. $x^2\tan(x^2+x+1)$.

13. $\frac{x^2}{1+\cos x}+\tan\frac{x}{2}$. 14. $x\arcsin x$. 15. $\sin(2\arctan x)$. 16. $\frac{x}{1+x^2}-\arctan x$.

17. $\arcsin(\sin x)$. 18. $\arctan\frac{1-\cos x}{\sin x}$. 19. $\arctan\left(x-\sqrt{1+x^2}\right)$. 20. $x(\log x-1)$.

21. $\log\left(\tan\frac{x}{2}\right)$. 22. $\log\log x$. 23. $x\tan x+\log(\cos x)$. 24. $\arctan\left(\log\frac{1}{x^2}\right)$.

25. $e^{e^x}$. 26. $xe^{1-\cos x}$. 27. $2^{\frac{x}{\log x}}$. 28. $\log(\sinh x-1)$.

29. $\cosh(\sinh x)$. 30. $\frac{1}{\sqrt{1+e^{-\sqrt{x}}}}$. 31. $\sqrt{\arctan\left(\sinh\frac{x}{3}\right)}$. 32. $x^x$.

33. $x^{x^x}$. 34. $(x^x)^x$. 35. $\sin\left(x^{\log x}\right)$. 36. $(\cos x)^{\frac{1}{x}}$.

EXERCISE 7.14.7. Given the following $f:[-1,1]\to\mathbb{R}$ find $a,b\in\mathbb{R}$ such that $\exists f'(0)$:

$$f(x)=\begin{cases}(a+1)\arcsin x-6(b+3)\sin x, & \text{if } -1\leqslant x\leqslant 0,\\[2ex] 2a(x^4+x)-(b+3)(\sqrt{x}+\tan x), & \text{if } 0<x\leqslant 1,\end{cases}$$

EXERCISE 7.14.8. Given the following $f:[-1,1]\to\mathbb{R}$ find $a,b\in\mathbb{R}$ such that $\exists f'(0)$:

$$f(x)=\begin{cases}b(x^4+3x)+(2-a)\cos\frac{1}{x}, & \text{se } -1\leqslant x<0,\\[2ex] (2b+3)(e^x-1)+(2-a)\tan 3x, & \text{se } 0\leqslant x\leqslant 1,\end{cases}$$

Exercise 7.14.9. Determine $a \in \mathbb{R}$ such that

$$f(x) := \begin{cases} \frac{\cos \frac{\pi}{2x}}{\log x}, & 0 < x < 1,\ x > 1, \\ \\ a, & x = 1, \end{cases}$$

be continuous at $x = 1$. For such a value, determine if $f$ is differentiable at $x = 0$.

Exercise 7.14.10. Compute

1. $\displaystyle\lim_{x\to 0} \frac{2 - 2\cos x - (\sin x)^2}{x^4}$. 2. $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$. 3. $\displaystyle\lim_{x\to 0} \frac{\sin x - \log\cos x}{x \sin x}$. 4. $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$.

5. $\displaystyle\lim_{x\to +\infty} \frac{\cos(2x) - \cos x}{x^2}$. 6. $\displaystyle\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$. 7. $\displaystyle\lim_{x\to +\infty} \frac{x^3(\log x)^2}{e^x}$. 8. $\displaystyle\lim_{x\to +\infty} (e^x + 1)^{\frac{1}{x}}$.

9. $\displaystyle\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}$. 10. $\displaystyle\lim_{x\to 0+} \log_x(e^x - 1)$. 11. $\displaystyle\lim_{x\to 1} x^{\frac{1}{1-x}}$. 12. $\displaystyle\lim_{x\to 1} \left(\frac{1}{\log x} - \frac{1}{x-1}\right)$.

Exercise 7.14.11 ($\star$). Compute

1. $\displaystyle\lim_{x\to 0} \frac{e - (1+x)^{\frac{1}{x}}}{x}$. 2. $\displaystyle\lim_{x\to 1+} \frac{-x^x + x}{\log x - x + 1}$. 3. $\displaystyle\lim_{x\to 0+} \frac{\sqrt{x}\sin x + x^2}{\sqrt{e^x - 1}\log(x+1)}$.

4. $\displaystyle\lim_{x\to 0+} \left(\frac{1}{1-\cos x} - \frac{2}{x^2}\right)$. 5. $\displaystyle\lim_{x\to 0} (1 + x^2)^{\frac{1}{(\sin x)^2}}$. 6. $\displaystyle\lim_{x\to 0} \left(\frac{1 - \log(x+1)}{1 + \sin x}\right)^{\frac{1}{x}}$.

7. $\displaystyle\lim_{x\to +\infty} \left(x^2 \log \frac{x}{x+1} + x\right)$. 8. $\displaystyle\lim_{x\to 0+} \frac{(1+x)^{\frac{2}{x}} - e^2}{x}$. 9. $\displaystyle\lim_{x\to +\infty} x \left[\left(\frac{2x+1}{2x}\right)^{6x} - e^3\right]$.

Exercise 7.14.12. For any of the following functions find: domain, eventual symmetries, behavior at the boundary of the domain (including eventual asymptotes), sign (if possible), continuity, differentiability, limits of $f'$ at the boundaries of the domain of $f'$, sign of $f'$, monotonicity, local and global extreme points, convexity and flexes. Finally, plot the graph of the function.

(1) $f(x) := x^5 + x^4 - 2x^3$.
(2) $f(x) := x^2 \log|x|$.
(3) $f(x) := x\sqrt{\frac{x-1}{x+1}}$.
(4) $f(x) := \log\left|1 - \frac{1}{\log|x|}\right|$.
(5) $f(x) := \arcsin\left(2x\sqrt{1-x^2}\right)$.
(6) $f(x) := 2x + \arctan \frac{x}{x^2-1}$.
(7) $f(x) := 2x + \frac{\sinh x}{\sinh x - 1}$.
(8) $f(x) := x\sqrt{|\log x|}$.
(9) $f(x) := x^x$.

EXERCISE 7.14.13. For each of the following functions find: domain, sign (only for numbers 6,7,8,9), limits and asymptotes at the endpoints of the domain, continuity and differentiability, compute $f'$, limits of $f'$ at the boundary of its domain, eventual points where $f$ could be extended continuously and with derivative, monotonicity, extreme points and plot a graph:

1. $\arctan\left|\dfrac{1}{\sinh x}\right| - \dfrac{\sinh x}{3}$. 2. $\arctan(e^x - 1) + \log|e^x - 4|$. 3. $\arctan\left((x+1)e^{\frac{1}{x}}\right)$.

4. $2x + \arctan\dfrac{x}{x^2-1}$. 5. $2\arcsin\dfrac{1}{\cosh(x-1)} + x$. 6. $\dfrac{(x+1)^{1/3}}{\log^3(x+1)}$

7. $\sqrt{3}x - \sqrt{3x^2+2x}$ 8. $\log\left(\sinh^2 x + 2\sinh x + 4\right)$ 9. $\arcsin\sqrt{1-(\log x)^2}$.

EXERCISE 7.14.14. Study the following inequalities

1. $\log x < \sqrt{x}$. 2. $\log x - \dfrac{x-1}{x} \geqslant 0$, 3. $e^x \geqslant 1 + x + \dfrac{x^2}{2}$. 4. $\dfrac{x - 2\log|x+1|}{\sqrt{e^x - 2x - 1}} \leqslant 0$.

EXERCISE 7.14.15 (★). Study in dependence of the parameter $\alpha > 0$, solutions of

$$e^y = \alpha y,\ y \in \mathbb{R},$$

EXERCISE 7.14.16 (★). Find all possible values $\alpha \in \mathbb{R}$ such that the following function be monotone increasing:

$$f(x) := \alpha x - \frac{x^3}{1+x^2}.$$

EXERCISE 7.14.17 (★★). Find all possible values $\alpha \in \mathbb{R}$ such that the function $f_\alpha(x) := e^{\alpha x} - \alpha^2 x$ be monotone on $[0, +\infty[$.

EXERCISE 7.14.18 (★). Let $p > 1$. Prove the inequality

$$2^{1-p} \leqslant \frac{t^p + 1}{(t+1)^p} \leqslant 1,\ \forall t \geqslant 1.$$

Deduce by this the inequality

$$2^{1-p} \leqslant \frac{x^p + y^p}{(x+y)^p} \leqslant 1,\ \ \forall x, y > 0.$$

EXERCISE 7.14.19 (★★). For which value $\alpha \in \mathbb{R}$ the function $f_\alpha(x) := e^x - \alpha x^3$ is convex on $\mathbb{R}$?

EXERCISE 7.14.20. Compute

1. $\displaystyle\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$.

2. $\displaystyle\lim_{x\to 0} \frac{x\cos x - \sin x}{1 + x^2 - e^{x^2} + (\sin x)^3}$.

3. $\displaystyle\lim_{x\to +\infty} \left(x - x^2 \log\left(1 + \frac{1}{x}\right)\right)$.

4. $\displaystyle\lim_{x\to 0} \frac{x^3 + x^2(\sin x)^2 + \sin x^2}{x^4 + x^3 + x\sin x}$.

5. $\displaystyle\lim_{x\to 0} \frac{\log(1+x^2) + x^2 + (\tan x)^2 + \sin x}{x^3 + \log(1+x)}$.

6. $\displaystyle\lim_{x\to 0+} \frac{(\sin x)^3 + x^{\frac{2}{3}}}{\sqrt{1-\cos x} + (\tan x)^2 + \arctan x}$.

7. $\displaystyle\lim_{x\to 0} \frac{\log(2 - \cos(2x))}{(\log(\sin(3x)+1))^2}$.

8. $\displaystyle\lim_{x\to 0} \frac{e^{\sin x} - 1 - x}{x^2}$.

9. $\displaystyle\lim_{x\to 0} \frac{e^{(\sin x)^3} - 1 - (\tan x)^3}{x^3\left(e^{x^2} - e^{(\sin x)^2}\right)}$.

10. $\displaystyle\lim_{x\to 0} \frac{\log(1 + x\arctan x) - e^{x^2} + 1}{\sqrt{1+2x^4} - 1}$.

11. $\displaystyle\lim_{x\to 0} \frac{\log(1+\sin x) - x + \frac{x^2}{2}}{(\tan x)^3 + x^5}$.

12. $\displaystyle\lim_{x\to 0+} \frac{xe^{x^7} - \cos(x^4) + 1 - x}{\sinh x^4 - \log(1+x^4)}$.

13. $\displaystyle\lim_{x\to 0+} \frac{e^{\frac{x^2}{2}} - \cosh x + (\sinh x)^3}{x\log\cosh x + (x\log x)^4}$.

EXERCISE 7.14.21. Compute, in dependence of $\alpha > 0$, the following limits:

1. $\displaystyle\lim_{x\to 0} \frac{\log(\cos x)}{x^\alpha}$.

2. $\displaystyle\lim_{x\to 0^+} \frac{\ln(1+x^\alpha) - \sin x}{1 - \cos(x^\alpha)}$.

3. $\displaystyle\lim_{x\to 0+} \frac{e^{x^\alpha} - 1 - x^\alpha - \frac{1}{2}\sin x}{\ln(1+x) - x}$.

4. $\displaystyle\lim_{x\to 0+} \frac{\cos(x^\alpha) - e^x}{x\log(1+x^\alpha) - x^\alpha}$.

5. $\displaystyle\lim_{x\to 0^+} \frac{x^2 - \arctan(x^2)}{e^{x^2} - \cos\left(x^{2\alpha}\right)}$.

6. $\displaystyle\lim_{x\to 0^+} \frac{e^{x^\alpha} - 1 + x\log x}{\sin\left(x^{2\alpha}\right) + 1 - \cos\left(x^2\right)}$.

7. $\displaystyle\lim_{x\to 0+} \frac{\sin(x^\alpha) - \sinh(x^2)}{\log(1+2x^2) - (\cos(2x) - 1)}$.

8. $\displaystyle\lim_{x\to 0+} \frac{\cos x - e^{-x^2}}{\cosh x + \cos(x^\alpha) - 2}$.

9. $\displaystyle\lim_{x\to 0+} \frac{\log(1+x^3) - e^{x^\alpha} - 1}{x\sinh(x^2) - \sin x(\cos x - 1}$

10. $\displaystyle\lim_{x\to 0+} \frac{xe^{x^3} - \cos(x^2) + 1 - x}{\sinh x^\alpha - \log(1+x^4)}$.

11. $\displaystyle\lim_{x\to 0+} \frac{e^{-\alpha x^2} - \cos x + (\log(1+x))^2}{x^3}$.

EXERCISE 7.14.22 (★). Compute

$$\lim_{x\to 0+} \frac{\sqrt[3]{1 - \sin\sqrt{x}} - e^x + \frac{\sqrt{x}}{3}}{\arctan x}.$$

EXERCISE 7.14.23 (★). Find $\alpha > 0$ (if they exist) such that the following limit is finite and different by 0:

$$\lim_{x\to 0+} \frac{\log\left(x+e^{\sqrt{x}}\right) - \sin\sqrt{x} - x}{x^\alpha}.$$

EXERCISE 7.14.24. Find $a, b \in \mathbb{R}$ such that the following limit exists finite and not 0:

$$\lim_{x\to 0} \frac{a\sin x - 2b\log(1+x) + \frac{2}{3}(a-2)x^3}{x^2}$$

EXERCISE 7.14.25 (★). Compute

$$\lim_{x\to 0}(\sin x)^{-2}\left(2^{\sqrt[3]{1+\sin x}} - 2 - \frac{2\log 2}{3}\sin x\right).$$

EXERCISE 7.14.26. Discuss convergence for each of the following series:

1. $\displaystyle\sum_{n=1}^{+\infty} \sin\frac{1}{n}$. 2. $\displaystyle\sum_{n=1}^{+\infty}\left(1+e^{\frac{1}{n}} - 2e^{\frac{1}{2n}}\right)$. 3. $\displaystyle\sum_{n=1}^{+\infty}\sqrt{n}\log\left(\frac{2n^2+3}{2n^2+2}\right)$.

4. $\displaystyle\sum_{n=1}^{+\infty}\frac{\log(n+1)-\log n}{\sqrt{n}+\log n}$. 5. $\displaystyle\sum_{n=1}^{+\infty}\left(e^{\frac{1}{\sqrt{n}}}-1\right)\log\left(1+\frac{1}{\sqrt{n}}\right)$. 6. $\displaystyle\sum_{n=1}^{\infty}\left(e^{\frac{1}{\sqrt{n}}}-1\right)\left(\sin\frac{1}{n}-\frac{1}{n}\right)$.

EXERCISE 7.14.27. Determine for which $\alpha > 0$ the following series are convergent:

1. $\displaystyle\sum_{n=1}^{+\infty} n\left(\cos\frac{\sqrt{2}}{n} - e^{-\frac{1}{n^\alpha}}\right)$. 2. $\displaystyle\sum_{n=1}^{\infty}(n+\sin n)\left(\frac{1}{n^\alpha} - \sin\frac{1}{n^\alpha}\right)$.

3. $\displaystyle\sum_{n=1}^{\infty}\left(\log\left(\cos\frac{1}{n}\right) - \frac{1}{n^\alpha}\right)$. 4. $\displaystyle\sum_{n=1}^{+\infty} n\left(e^{\frac{1}{2n^\alpha}} - \cosh\frac{1}{n}\right)$.

5. $\displaystyle\sum_{n=1}^{+\infty}\sqrt{n}\left(\sinh\frac{1}{n} - \log\left(1+\frac{1}{n^\alpha}\right)\right)$. 6. $\displaystyle\sum_{n=1}^{+\infty} n^2\left(\sin\frac{1}{n^\alpha} - \sinh\frac{1}{n^3}\right)$.

7. $\displaystyle\sum_{n=1}^{+\infty} n^3\left(\sin\frac{1}{n^4} - e^{\frac{1}{n^\alpha}} + 1\right)$. 8. $\displaystyle\sum_{n=1}^{\infty} n^\alpha\left(\frac{1}{n} - \arctan\frac{1}{n}\right)$

EXERCISE 7.14.28 (★). Find $\alpha, \beta \in \mathbb{R}$ such that the following series is convergent:

$$\sum_{n=0}^{\infty} n^\alpha\left[e^{\frac{\beta}{n^2}} - \cos\frac{1}{n} + \left(\log\left(1+\frac{1}{n}\right)\right)^2\right].$$

EXERCISE 7.14.29 (★). Discuss simple and absolute convergence for

1. $\displaystyle\sum_{n=1}^{+\infty}\sin\frac{(-1)^n}{n}$. 2. $\displaystyle\sum_{n=1}^{+\infty}\log\left(1+\frac{(-1)^n}{n}\right)$. 3. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\left(1+\frac{1}{n}\right)^n - e\right)$. 4. $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{1}{n+(-1)^n}$

Exercise 7.14.30 (★★). Let $a > 0$. Discuss convergence for

$$\sum_{n=1}^{\infty} \sin\left(\pi\sqrt{n^2 + a^2}\right).$$

Exercise 7.14.31. Find the radius $r$ and height $h$ of a cylindrical can with fixed surface $S$ and volume maximum as possible.

Exercise 7.14.32. Among all rectangles inscribed in a circumference of radius $r$, find those with maximum area.

Exercise 7.14.33. Find the maximum area for an isosceles triangle inscribed in a circumference of radius $r$.

Exercise 7.14.34. Find the minimum surface cone with circular base and height perpendicular to the base inscribed into a sphere of radius $r$.

Exercise 7.14.35. Consider the part of the parabola $y = -x^2 + 4$ in the first quarter. The tangent to the parabola at some point forms a triangle with axes $x$ and $y$. Find those with minimum and maximum area (if they exists).

Exercise 7.14.36. Among all convex polygons with vertexes on a circumference of radius $r$, find (if they exist) those with maximum perimeter.

Exercise 7.14.37. A cylindrical can with base radius $r$ and height $h$ costs $C$ per $cm^2$ of aluminium and $C/3$ per $cm^2$ to paint the lateral surface. Determine, in dependence of $C$ and of the volume $V$ fixed, $r$ and $h$ such that the cost of the can is minimum.

Exercise 7.14.38. A plane course connects two airports $A$ and $B$. The plane takes off by $A$ ascending along a straight line up to height $h$, then it flight at speed $v_C$ up the beginning of the descent to $B$. If $p \in ]-1, 1[$ is the slope of the ascent/descent the speed of the plane is $v(p) = v_C(1 - p)$. Find $p$ in such a way that the flight is shorter as possible.

Exercise 7.14.39. A sail boat goes back up on a regatta field of length $\ell$. The wind has intensity $F$ and a constant direction, opposite to the direction of the boat. If $\theta$ is the angle between the boat course and the wind direction, the pressure on the sail is $F \sin\theta$. We suppose that the course is done by two straight parts and the initial angle with the wind is $\alpha$. Determine the time the boat has to turn in such a way that the total time is minimum. Which is the angle $\alpha$ that minimizes the total time?

Exercise 7.14.40. What is the maximum volume of a box that can be constructed from a suitably plied piece of cardboard, which is cut as a unique piece from a square cardboard of side $L$?

CHAPTER 8

# Integral Calculus

## 8.1. Area of a plane figure

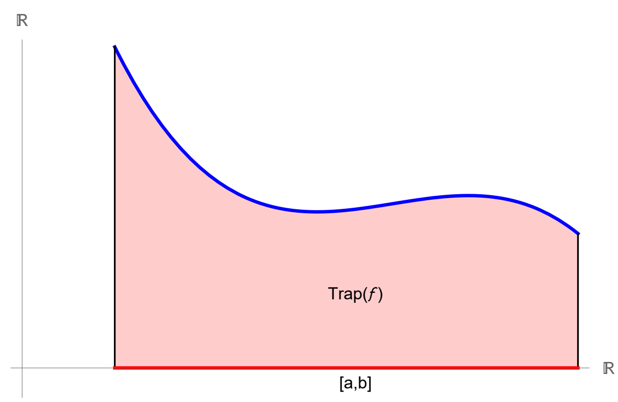
The main goal of *Integral Calculus* is to find a general method to compute the areas of plane figures. Actually, the range of applications goes much beyond the geometric side of the story, providing a base for modern Probability, a precise definition of fundamental physical entities, and to many other applied problems.

The concept of area goes back to the origin of Mathematics. Area is a quantity we associate with a plane figure to measure its *size*. For elementary figures, as rectangles, this number is easily defined: $A = b \cdot h$ where $b$ is the length of the base, $h$ is the length of height. By this formula, we have a formula of area for a triangle, hence, in principle, for *polygons*. The difficulty increases when we pretend to measure the size of a generic plane figure. For certain particular figures such as circles or ellipses, Greeks found formulas, but they were not able to go too far.

A revolution arrived only centuries later when Torricelli proposed the method that would have been formalized by Riemann a few centuries later: *to compute the area of a figure A we fill the figure by non overlapping rectangles; the area of A will be the (possibly infinite) sum of the areas of the rectangles used to tile A*.

The first step is to give a mathematical form to the kind of sets we consider to define their area. A subset $S$ of the plane $xy$ may appear as too general. We will consider here the following description. Let $f : [a, b] \longrightarrow [0, +\infty[$. We use $f$ as "upper boundary" of a plane region, called **trapezoid**, precisely defined as

$$\text{Trap}(f) := \{(x, y) \ : \ a \leqslant x \leqslant b, \ 0 \leqslant y \leqslant f(x)\} .$$

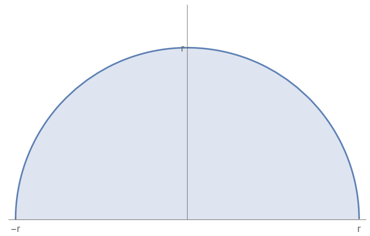



The goal is to define the area of Trap($f$).

At first sight this may appear too restrictive. For example, even a simple plane figure as a circle cannot be described as a trapezoid. However, this problem can be easily circumvented considering a half circle, for instance

$$\{(x, y) \ : \ y \geqslant 0, \ x^2 + y^2 \leqslant r^2\} = \{(x, y) \ : \ -r \leqslant x \leqslant r, \ 0 \leqslant y \leqslant \sqrt{r^2 - x^2}\} = \text{Trap}(f),$$

where $f : [-r, r] \longrightarrow [0, +\infty[$ is $f(x) = \sqrt{r^2 - x^2}$.



We now set the basis for the rigorous foundation of the Torricelli method:

**Definition 8.1.1**

A **subdivision** of an interval $[a, b]$ is a finite set of points $\pi := \{x_0, x_1, \dots, x_N\}$ such that

$$a = x_0 < x_1 < \dots < x_N = b.$$

We set

$$|\pi| := \max_{k=0,\dots,N-1} |x_{k+1} - x_k|.$$

The set of all the subdivisions of $[a, b]$ will be denoted by $\Pi[a, b]$.

**Definition 8.1.2**

Let $f : [a, b] \longrightarrow [0, +\infty[$ be a **bounded** function. If $\pi \in \Pi[a, b]$ we call **inferior sum of $f$ over $\pi$** the quantity

$$\underline{S}(\pi) := \sum_{k=0}^{N-1} m_k (x_{k+1} - x_k), \quad m_k := \inf_{x \in [x_k, x_{k+1}]} f(x).$$

Similarly, the **superior sum of $f$ over $\pi$** is

$$\overline{S}(\pi) := \sum_{k=0}^{N-1} M_k (x_{k+1} - x_k), \quad M_k := \sup_{x \in [x_k, x_{k+1}]} f(x).$$

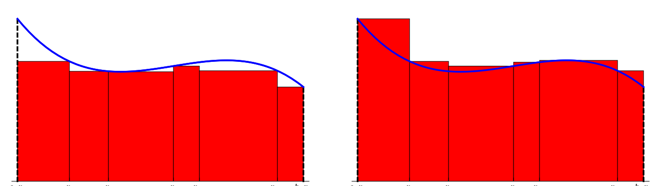


FIGURE 1. At left, the area of an inferior sum, at right the same for a superior sum

By construction, the inferior and superior sums are, respectively, approximations by defect/excess of the area of Trap$(f)$. These sums fulfil simple and natural properties (proofs are omitted: they are not particularly difficult and they do not add any insight):

**Proposition 8.1.3**

Let $f : [a,b] \longrightarrow [0,+\infty[$ be a bounded function. Then

i) $\underline{S}(\pi) \leqslant \overline{S}(\pi)$, $\forall \pi \in \Pi[a,b]$;

ii) $\underline{S}(\pi_1) \leqslant \underline{S}(\pi_2)$, $\overline{S}(\pi_2) \leqslant \overline{S}(\pi_1)$, $\forall \pi_1, \pi_2 \in \Pi[a,b]$ with $\pi_1 \subset \pi_2$.

The next step is to define the best approximations by defects and by excess of the area of Trap$(f)$:

**Definition 8.1.4**

Let $f : [a,b] \longrightarrow [0,+\infty[$ be bounded. We call **inferior area** and **superior area**

$$\underline{A}(f) := \sup_{\pi \in \Pi[a,b]} \underline{S}(\pi), \quad \overline{A}(f) := \inf_{\pi \in \Pi[a,b]} \overline{S}(\pi).$$

Clearly

**Proposition 8.1.5**

Let $f : [a,b] \longrightarrow [0,+\infty[$ be bounded. Then

$$-\infty \leqslant \underline{A}(f) \leqslant \overline{A}(f) \leqslant +\infty. \tag{8.1.1}$$

A question arises now: is it possible to have $\underline{A}(f) < \overline{A}(f)$? The answer is yes, as the following example shows:

Example 8.1.6 (Dirichlet function). *Let*

$$f(x) := \begin{cases} 0, & x \in \mathbb{Q} \cap [0,1], \\ \\ 1, & x \in \mathbb{Q}^c \cap [0,1], \end{cases}$$

*(Dirichlet function). Then* $\underline{A}(f) = 0 < 1 = \overline{A}(f)$.

Sol. — Let $\pi = \{x_0, x_1, \dots, x_N\}$ be a subdivision of $[0,1]$. Because of the density of rationals/irrationals in the real line

$$m_k = \inf_{x \in [x_k, x_{k+1}]} f(x) = 0, \quad M_k := \sup_{x \in [x_k, x_{k+1}]} f(x) = 1.$$

Therefore

$$\underline{S}(\pi) = \sum_{k=0}^{N-1} m_k (x_{k+1} - x_k) = 0, \quad \overline{S}(\pi) = \sum_{k=0}^{N-1} M_k (x_{k+1} - x_k) = \sum_{k=0}^{N-1} (x_{k+1} - x_k) = 1.$$

Hence

$$\underline{A}(f) = \sup_{\pi \in \Pi[0,1]} \underline{S}(\pi) = 0, \quad \overline{A}(f) = \inf_{\pi \in \Pi[0,1]} \overline{S}(\pi) = 1. \quad \square$$

When the best approximation by defect coincides with the best approximation by excess, we call the common value *area of* Trap$(f)$:

**Definition 8.1.7**

Let $f : [a, b] \longrightarrow [0, +\infty[$ be a bounded function. If $\underline{A}(f) = \overline{A}(f) \in \mathbb{R}$ we call **area of** Trap$(f)$ the quantity

$$A(f) := \underline{A}(f) \equiv \overline{A}(f).$$

Example 8.1.8. Let us check that the definition works coherently with the area of rectangles, that is, *the area of a rectangle of base $b$ and height $h$ is $b \cdot h$.*

Sol. — Setting $f : [0, b] \longrightarrow [0, +\infty[$ $f(x) \equiv h$, we have

$$\text{Trap}(f) = \{(x, y) : 0 \leqslant x \leqslant b, \ 0 \leqslant y \leqslant h\},$$

is a rectangle of base $b$ and height $h$. Let $\pi \in \Pi[0, b]$,

$$\underline{S}(\pi) = \sum_k m_k(x_{k+1} - x_k) = \sum_k b(x_{k+1} - x_k) = hb$$

and simlarly $\overline{S}(\pi) = hb$, hence $\underline{A}(f) = \overline{A}(f) = hb$. $\square$


The following is a characterization for the existence of the area $A(f)$:

**Proposition 8.1.9**

Let $f : [a, b] \longrightarrow [0, +\infty[$ a bounded function. Then, $A(f)$ is well defined iff

$$\forall \varepsilon > 0, \ \exists \pi_\varepsilon \in \Pi[a, b] \ : \ \overline{S}(\pi_\varepsilon) - \underline{S}(\pi_\varepsilon) \leqslant \varepsilon. \tag{8.1.2}$$

Proof. Necessity: by the definition of inf and sup,

$$\forall \varepsilon > 0, \ \exists \widehat{\pi}_\varepsilon, \ \widetilde{\pi}_\varepsilon \in \Pi[a, b] \ : \ \underline{S}(\widehat{\pi}_\varepsilon) \geqslant \underline{A}(f) - \frac{\varepsilon}{2}, \quad \overline{S}(\widetilde{\pi}_\varepsilon) \leqslant \overline{A}(f) + \frac{\varepsilon}{2}.$$

If $A(f) = \underline{A}(f) = \overline{A}(f)$ then

$$\overline{S}(\widetilde{\pi}_\varepsilon) - \underline{S}(\widehat{\pi}_\varepsilon) \leqslant \overline{A}(f) + \frac{\varepsilon}{2} - \underline{A}(f) + \frac{\varepsilon}{2} \leqslant \varepsilon.$$

Setting $\pi_\varepsilon = \widehat{\pi}_\varepsilon \cup \widetilde{\pi}_\varepsilon$, by Proposition 8.1 we have $\overline{S}(\pi_\varepsilon) \leqslant \overline{S}(\widetilde{\pi}_\varepsilon)$ and $\underline{S}(\pi_\varepsilon) \geqslant \underline{S}(\widehat{\pi}_\varepsilon)$, so that

$$\overline{S}(\widetilde{\pi}_\varepsilon) - \underline{S}(\widehat{\pi}_\varepsilon) \leqslant \overline{S}(\pi_\varepsilon) - \underline{S}(\pi_\varepsilon) \leqslant \varepsilon,$$

from which the conclusion follows.

Sufficiency: suppose (8.1.2) holds. Since $\underline{S}(\pi_\varepsilon) \leqslant \underline{A}(f) \leqslant \overline{A}(f) \leqslant \overline{S}(\pi_\varepsilon)$, we obtain

$$0 \leqslant \overline{A}(f) - \underline{A}(f) \leqslant \overline{S}(\pi_\varepsilon) - \underline{S}(\pi_\varepsilon) \leqslant \varepsilon,$$

and being $\varepsilon > 0$ arbitrary we conclude that $0 \leqslant \overline{A}(f) - \underline{A}(f) \leqslant 0$, that is, $\underline{A}(f) = \overline{A}(f)$. $\square$

In particular, we get

Corollary 8.1.10. *Let $f : [a,b] \longrightarrow [0,+\infty[$ a bounded function. Then, $A(f)$ is well defined iff*

$$\exists (\pi_n)_{n\in\mathbb{N}} \subset \Pi[a,b] \ : \ \lim_n \underline{S}(\pi_n) = \lim_n \overline{S}(\pi_n).$$

*The common limit is $A(f)$.*

Proof. Necessity: Since $\overline{A}(f) = \underline{A}(f) = A(f)$, for every $n \in \mathbb{N}$ there exists two subdivisions $\pi_n^+, \pi_n^-$ such that

$$A(f) - \frac{1}{n} \leqslant \underline{S}(\pi_n^-) \leqslant A(f) \leqslant \underline{S}(\pi_n^+) \leq A(f) + \frac{1}{n}.$$

For every $n \in \mathbb{N}$ consider the subdivision $\pi_n := \pi_n^- \cup \pi_n^+$ and observe that

$$A(f) - \frac{1}{n} \leqslant \underline{S}(\pi_n^-) \leqslant \underline{S}(\pi_n) \leqslant A(f) \leqslant \overline{S}(\pi_n) \leqslant \underline{S}(\pi_n^+) \leq A(f) + \frac{1}{n}.$$

Therefore, we get

$$\lim_n \underline{S}(\pi_n) = \lim_n \overline{S}(\pi_n) = A(f)$$

Sufficiency We are assuming that

$$\exists (\pi_n)_{n\in\mathbb{N}} \subset \Pi[a,b] \ : \ \lim_n \underline{S}(\pi_n) = \lim_n \overline{S}(\pi_n).$$

Therefore, for every $\varepsilon > 0$ there exists $N$ such that $\overline{S}(\pi_n) - \underline{S}(\pi_n) \leqslant \varepsilon$ for all $n \in \mathbb{N}$, so that by the previous proposition we deduce that $A(f)$ is well defined (i.e. $\overline{A}(f) = \underline{A}(f) = A(f)$). □

We apply this corollary to some simple but non-trivial calculations of area.

Example 8.1.11. *The area of a rectangle triangle of base $b$ and height $h$ is $\frac{1}{2}b \cdot h$.*

Sol. — We may describe the triangle as Trap$(f)$ where $f : [0,b] \longrightarrow [0,+\infty[$, $f(x) = \frac{h}{b}x$. Let's check that $A(f) = \frac{1}{2}b \cdot h$.

To this aim, let $\pi_n$ be the subdivision of $[0,b]$ in $n$ equal parts, that is

$$\pi_n = \left\{0, \frac{b}{n}, 2\frac{b}{n}, \dots, k\frac{b}{n}, \dots, n\frac{b}{n} = b\right\}, \quad x_k = k\frac{b}{n}, \ k = 0,1,2,\dots,n.$$

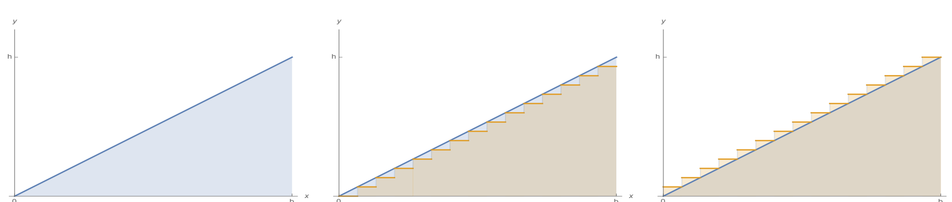


Figure 2. Left to right: $A(f)$, $\underline{S}(\pi_n)$, $\overline{S}(\pi_n)$

Then

$$m_k = \inf_{x\in[k\frac{b}{n},(k+1)\frac{b}{n}]} \frac{h}{b}x = \frac{h}{b}k\frac{b}{n} = \frac{h}{n}k, \quad M_k = \sup_{x\in[k\frac{b}{n},(k+1)\frac{b}{n}]} \frac{h}{b}x = \frac{h}{n}(k+1).$$

Thus

$$\underline{S}(\pi_n) = \sum_{k=0}^{n-1} m_k (x_{k+1} - x_k) = \sum_{k=0}^{n-1} k \frac{h}{n}\frac{b}{n} = \frac{hb}{n^2}\sum_{k=0}^{n-1} k, \quad \overline{S}(\pi_n) = \frac{hb}{n^2}\sum_{k=0}^{n-1}(k+1) = \frac{hb}{n^2}\sum_{k=1}^{n} k.$$

Now, it is well known that

$$\sum_{k=0}^{N} k = \frac{N(N+1)}{2},$$

by which

$$\underline{S}(\pi_n) = \frac{hb}{n^2}\frac{(n-1)n}{2} \longrightarrow \frac{hb}{2}, \quad \overline{S}(\pi_n) = \frac{hb}{n^2}\frac{n(n+1)}{2} \longrightarrow \frac{hb}{2}. \quad \square$$

Example 8.1.12 (Archimedes). *Compute the area of a parabolic segment $y = ax^2$, $x \in [0, b]$, with $a, b > 0$. We have*

$$A(f) = a\frac{b^3}{3}.$$

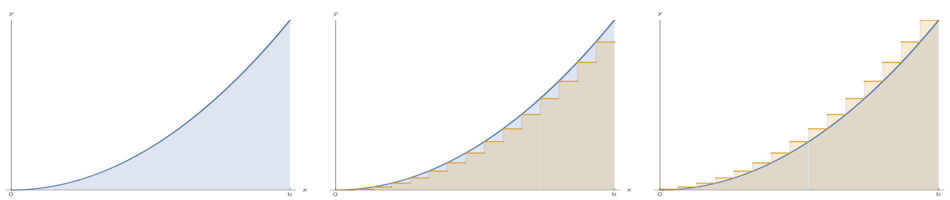


Figure 3. Left to right, $A(f)$, $\underline{S}(\pi_n)$, $\overline{S}(\pi_n)$

Sol. — As in the previous example, let $\pi_n$ the subdivision of $[0, b]$ in $n$ equal parts. Since $x^2 \nearrow$ on $[0, b]$,

$$m_k = \inf_{x \in [k\frac{b}{n}, (k+1)\frac{b}{n}]} ax^2 = a\left(k\frac{b}{n}\right)^2 = a\frac{b^2}{n^2}k^2, \quad M_k = \sup_{x \in [k\frac{b}{n}, (k+1)\frac{b}{n}]} ax^2 = a\left((k+1)\frac{b}{n}\right)^2 = a\frac{b^2}{n^2}(k+1)^2,$$

Therefore

$$\underline{S}(\pi_n) = \sum_{k=0}^{n-1} a\frac{b^2}{n^2}k^2\frac{b}{n} = a\frac{b^3}{n^3}\sum_{k=0}^{n-1} k^2, \quad \overline{S}(\pi_n) = a\frac{b^3}{n^3}\sum_{k=0}^{n-1}(k+1)^2 = a\frac{b^3}{n^3}\sum_{k=1}^{n} k^2.$$

Now, it is well known (we accept here) that

$$\sum_{k=0}^{N} k^2 = \frac{N(N+1)(2N+1)}{6},$$

by which

$$\underline{S}(\pi_n) = a\frac{b^3}{n^3}\frac{(n-1)n(2(n-1)+1)}{6} \longrightarrow \frac{b^3}{3}, \quad \overline{S}(\pi_n) = a\frac{b^3}{n^3}\frac{n(n+1)(2n+1)}{6} \longrightarrow a\frac{b^3}{3}. \quad \square$$

The last example is interesting because it cannot be computed by reducing to some area of elementary figures, and indeed it was one of the greatest achievements of Archimedes. However, the procedure seems to be based on extremely particular calculations, hardly repeatable in general. Thus, we look for some simple and general condition that ensures the existence of area for a large class of functions. Fortunately, we have general results, the following is one of the most relevant:

**Theorem 8.1.13**

Let $f \in \mathscr{C}([a,b])$, $f \geqslant 0$ on $[a,b]$. Then $A(f)$ is well defined.

Proof. The actual proof would require an use of *uniform continuity*. Since in these notes we are not going to discuss this notion, we settle for the proof of a weaker statement, namely we strength the hypothesis $f \in \mathscr{C}([a,b])$ into the stronger assumption $f \in \mathscr{C}^1([a,b])$. [1] Let $\pi_n$ the subdivision of $[a,b]$ in $n$ equal parts, that is

$$\pi_n = \{x_0, x_1, \dots, x_n\}, \quad x_k = a + k\frac{b-a}{n}, \quad k = 0,1,2,\dots,n.$$

Since $x_{k+1} - x_k = \frac{b-a}{n}$,

$$\underline{S}(\pi_n) = \sum_{k=0}^{n-1} m_k (x_{k+1} - x_k) = \frac{b-a}{n}\sum_k m_k,$$

where, according to Weierstrass' theorem, $m_k = \inf_{[x_k,x_{k+1}]} f(x) \overset{f\in\mathscr{C}}{=} \min_{[x_k,x_{k+1}]} f(x) = f(\xi_k)$, for some $\xi_k \in [x_k, x_{k+1}]$. Similarly,

$$\overline{S}(\pi_n) = \frac{b-a}{n}\sum_{k=0}^{n-1} M_k,$$

where$M_k = \max_{[x_k,x_{k+1}]} f(x) = f(\eta_k)$ for a suitable $\eta_k \in [x_k, x_{k+1}]$. Then

$$\overline{S}(\pi_n) - \underline{S}(\pi_n) = \frac{b-a}{n}\sum_{k=0}^{n-1}(M_k - m_k) = \frac{b-a}{n}\sum_{k=0}^{n-1}\left(f(\eta_k) - f(\xi_k)\right).$$

Since we assumed $f$ derivable, according to Lagrange's theorem,

$$\frac{f(\eta_k) - f(\xi_k)}{\eta_k - \xi_k} = f'(c_k),$$

thus

$$\sum_{k=0}^{n-1}(f(\eta_k) - f(\xi_k)) = \sum_{k=0}^{n-1} f'(c_k)(\eta_k - \xi_k) \leqslant \sum_{k=0}^{n-1}|f'(c_k)||x_{k+1} - x_k|.$$

Since $f' \in \mathscr{C}([a,b])$, $|f'| \in \mathscr{C}([a,b])$ thus $|f'|$ is bounded, that is $|f'(y)| \leqslant C$ for every $y \in [a,b]$, thus

$$\overline{S}(\pi_n) - \underline{S}(\pi_n) \leqslant \frac{b-a}{n}\sum_{k=0}^{n-1} C\frac{b-a}{n} = C\left(\frac{b-a}{n}\right)^2\sum_{k=0}^{n-1} 1 = C\frac{(b-a)^2}{n^2}n = \frac{C(b-a)^2}{n} \longrightarrow 0,$$

for $n \longrightarrow +\infty$. Conclusion now follows by integrability test 8.1.2. □

However, let us remark that in order to be integrabl, a function does not need to be continuous. For example, it is possible to prove the

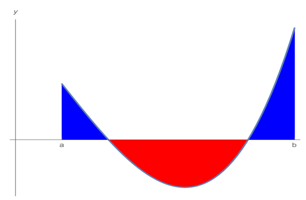
[1]Let us remind that $C^1([a,b])$ denotes the set of differentiable functions $f$ on $[a,b]$ whose derivative $f'$ are continuous on whole $[a,b]$. (At $x = a$ and $x = b$ the derivatives are identified with the right and left derivative, respectively.)

**Theorem 8.1.14**

Let $f : [a, b] \longrightarrow [0, +\infty[$ a monotonic function. Then $A(f)$ is well defined.

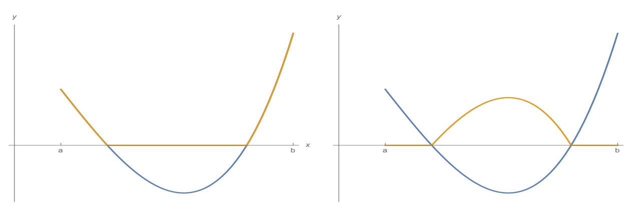
## 8.2. Integral

We introduce now the concept of *integral*. Let $f : [a, b] \longrightarrow \mathbb{R}$. The idea is very easy: when $f \geqslant 0$ we count positively the area of $f$, while when $f \leqslant 0$ we count negatively the area of $-f$.




To formalize the definition, we first introduce a useful notation. Given $f : [a, b] \longrightarrow \mathbb{R}$, we call $f_+, f_- : [a, b] \longrightarrow [0, +\infty[$ **positive/negative** part of $f$,

$$f_+(x) := \begin{cases} f(x), & f(x) \geqslant 0, \\ \\ 0, & f(x) < 0, \end{cases} \qquad f_-(x) := \begin{cases} -f(x), & f(x) \leqslant 0, \\ \\ 0, & f(x) > 0, \end{cases}$$




Remark 8.2.1. *Both $f_+, f_-$ are positive, easily $f = f_+ - f_-$ and $|f| = f_+ + f_-$.* □

With these definitions and respect to the above figure, $A(f_-)$ is the red area while $A(f_+)$ is the blue one.

**Definition 8.2.2**

Let $f : [a, b] \longrightarrow \mathbb{R}$ a bounded function. We say that $f$ is **Riemann integrable on** $[a, b]$ (notation $f \in \mathscr{R}[a, b]$) if $A(f_+), A(f_-)$ exist. In that case, we call **integral of** $f$ **on** $[a, b]$ the number

$$\int_a^b f(x)\, dx := A(f_+) - A(f_-).$$

Since for $f \in \mathscr{C}([a, b])$ both $f_\pm \in \mathscr{C}([a, b])$ we obtain

**Proposition 8.2.3**

Every continuous function $f$ on $[a,b]$ is integrable on $[a,b]$.

The main natural properties of the Riemann integral are summarized by the

**Proposition 8.2.4**

The following properties hold:

i) (**linearity**) if $f,g \in \mathscr{R}([a,b])$ then $\alpha f + \beta g \in \mathscr{R}([a,b])$, $\forall \alpha,\beta \in \mathbb{R}$ and

$$\int_a^b (\alpha f(x) + \beta g(x))\ dx = \alpha \int_a^b f(x)\ dx + \beta \int_a^b g(x)\ dx. \tag{8.2.1}$$

ii) (**monotonicity**) if $f,g \in \mathscr{R}([a,b])$ and $f \leqslant g$ on $[a,b]$ then

$$\int_a^b f(x)\ dx \leqslant \int_a^b g(x)\ dx. \tag{8.2.2}$$

In particular: if $f \geqslant 0$ on $[a,b]$ then $\int_a^b f(x)\ dx \geqslant 0$.

iii) (**triangular inequality**) if $f \in \mathscr{R}([a,b])$ then $|f| \in \mathscr{R}([a,b])$ and

$$\left|\int_a^b f(x)\ dx\right| \leqslant \int_a^b |f(x)|\ dx. \tag{8.2.3}$$

iv) (**decomposition**) if $f \in \mathscr{R}([a,b]), \mathscr{R}([b,c])$ then $f \in \mathscr{R}([a,c])$ and

$$\int_a^c f(x)\ dx = \int_a^b f(x)\ dx + \int_b^c f(x)\ dx.$$

An important property of the Riemann integral is the

**Theorem 8.2.5: integral mean theorem**

Let $f \in \mathscr{C}([a,b])$. There exists then $c \in [a,b]$ such that

$$\frac{1}{b-a}\int_a^b f(x)\ dx = f(c).$$

Proof. The proof is very easy: by Weierstrass thm $f$ has min/max over $[a,b]$, let's say

$$m = \min_{x\in[a,b]} f(x),\quad M = \max_{x\in[a,b]} f(x).$$

Then

$$m \leqslant f(x) \leqslant M,\ \forall x \in [a,b], \quad \overset{monotonicity}{\Longrightarrow} \quad \int_a^b m\ dx \leqslant \int_a^b f(x)\ dx \leqslant \int_a^b M\ dx,$$

that is

$$m(b-a) \leqslant \int_a^b f(x)\ dx \leqslant M(b-a), \quad \Longleftrightarrow \quad m \leqslant \frac{1}{b-a}\int_a^b f(x)\ dx \leqslant M.$$

By the intermediate values thm there exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx. \qquad \square$$

### 8.3. Fundamental Theorem of Integral Calculus

In this section, we will present the most important and deep results concerning Riemann integrals. For future convenience, it is useful to have defined the integral $\int_b^a f(x)\, dx$ for $f \in \mathscr{R}[a, b]$. Notice that $\int_b^a f$ is not a misprint: set

$$\int_b^a f := -\int_a^b f.$$

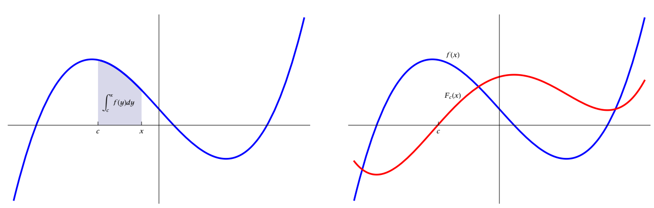
This position is fundamental to have a well-posed definition of the following concept:

**Definition 8.3.1**

Let $f \in \mathscr{R}([a, b])$ and $c \in [a, b]$ be fixed. We call **the integral function of $f$ centered at** $c$ the function

$$F_c : [a, b] \longrightarrow \mathbb{R}, \quad F_c(x) := \int_c^x f(y)\, dy, \ x \in [a, b].$$

An integral function is a natural object that measures the area (with sign) choosing $c$ as the point of area 0. To understand the connection between $F_c$ and $f$ let's consider a function (in blue in the next picture) and its integral function centered at some point $c$. First of all, $F_c(c) = \int_c^c f(y)\, dy = 0$. Moreover, increasing $x$ from $c$, being initially $f \geqslant 0$ we'll have $F_c \nearrow$.



We recognize that when $f \geqslant 0$, $F_c$ in increasing, while when $f \leqslant 0$, $F_c$ is decreasing. In other words:

$$F_c \nearrow \iff f \geqslant 0.$$

This is the relation that holds between $F_c$ and $F_c'$. Indeed, it turns out that

**Theorem 8.3.2**

Let $f \in \mathscr{C}([a, b])$. Then, any of its integral functions $F_c$ is a primitive of $f$, that is

$$F_c'(x) = f(x), \ \forall x \in [a, b], \ \forall c.$$

PROOF. Let $h > 0$ and notice that

$$\frac{F_c(x+h) - F_c(x)}{h} = \frac{1}{h}\left(\int_c^{x+h} f(y)\,dy - \int_c^x f(y)\,dy\right) = \frac{1}{h}\left(\int_c^x f(y)\,dy + \int_x^{x+h} f(y)\,dy - \int_c^x f(y)\,dy\right)$$

$$= \frac{1}{h}\int_x^{x+h} f(y)\,dy.$$

This is an integral mean (of $f$ over $[x, x+h]$): being $f$ continuous, by the mean value theorem there exists $\xi_h \in [x, x+h]$ such that

$$\frac{F_c(x+h) - F_c(x)}{h} = \frac{1}{h}\int_x^{x+h} f(y)\,dy = f(\xi_h).$$

As $h \longrightarrow 0+$, $\xi_h \longrightarrow x+$ hence, being $f$ continuous,

$$\lim_{h\to 0+} \frac{F_c(x+h) - F_c(x)}{h} = \lim_{h\to 0+} f(\xi_h) = f(x).$$

This shows that the right derivative of $F_c$ at $x$ is $f(x)$. Similarly, we can proceed with the left derivative, hence to conclude. $\square$

By the fundamental theorem of integral calculus, it follows a formula that connects the calculus of an integral with the calculus of primitives:

**Corollary 8.3.3: fundamental formula of integral calculus**

Let $f \in \mathscr{C}([a, b])$ and $F$ be any primitive of $f$ on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) =: F(x)\Big|_{x=a}^{x=b}. \tag{8.3.1}$$

PROOF. Indeed: take the integral function $F_a$. By the fundamental theorem of integral calculus, $F_a' = f$. Being $F$ also a primitive on $[a, b]$, $F_a - F$ is constant by Proposition 7.6, that is $F_a - F \equiv k \in \mathbb{R}$. Therefore

$$F_a(a) - F(a) = k, \quad F_a(b) - F(b) = k.$$

On the other hand

$$F_a(a) = \int_a^a f(x)\,dx = 0, \quad F_a(b) = \int_a^b f(x)\,dx,$$

so

$$\int_a^b f(x)\,dx = F_a(b) = F(b) + k = F(b) + (F_a(a) - F(a)) = F(b) - F(a). \quad \square$$

## 8.4. Integration formulas

Another way to review the (8.3.1) is the following: *if* $f \in \mathscr{C}^1([a, b])$ *then*

$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{8.4.1}$$

From this, some remarkable formulas follow:

**Proposition 8.4.1: integration by parts formula**

Let $f, g \in \mathscr{C}^1([a,b])$. Then

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_{x=a}^{x=b} - \int_a^b f'(x)g(x)\,dx. \tag{8.4.2}$$

Proof. Just notice that

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x), \implies \int_a^b (fg)'(x)\,dx = \int_a^b \left(f'(x)g(x) + f(x)g'(x)\right)\,dx.$$

By (8.4.1) we have

$$\int_a^b (fg)'(x)\,dx = f(x)g(x)\Big|_{x=a}^{x=b},$$

and now the conclusion follows. □

**Proposition 8.4.2: change of variable formula**

Let $f \in \mathscr{C}([a,b])$, $\psi : [c,d] \longrightarrow [a,b]$, $\psi \in \mathscr{C}^1$, bijection with inverse $\psi^{-1} \in \mathscr{C}^1$. Then

$$\int_a^b f(x)\,dx = \int_c^d f(\psi(y))|\psi'(y)|\,dy. \tag{8.4.3}$$

Proof. First, notice that $\psi' > 0$ or $\psi' < 0$ on $[c,d]$. Indeed: if $\psi'$ would change sign, then by continuity it should vanish somewhere. But being

$$\psi^{-1}(\psi(y)) = y,\ \forall y \in [c,d], \implies (\psi^{-1})'(\psi(y))\psi'(y) = 1,\ \forall y \in [c,d]$$

this is impossible. Suppose then that $\psi' > 0$ on $[c,d]$, in particular $\psi$ is strictly increasing. Then, if $F$ is a primitive of $f$

$$\begin{aligned}\int_c^d f(\psi(y))|\psi'(y)|\,dy &= \int_c^d f(\psi(y))\psi'(y)\,dy = \int_c^d F'(\psi(y))\psi'(y)\,dy = \int_c^d (F(\psi(y)))'\,dy \\ &= F(\psi(d)) - F(\psi(c)) = F(b) - F(a) = \int_a^b F'(x)\,dx = \int_a^b f(x)\,dx. \quad \square\end{aligned}$$

Example 8.4.3 (Area of a disk). *Compute the area of a disk of radius $r$.*

Sol. — Clearly

$$\text{Area}\left(x^2 + y^2 \leqslant r^2\right) = 2\int_{-r}^r \sqrt{r^2 - x^2}\,dx.$$

The function $f(x) := \sqrt{r^2 - x^2}$ is continuous on $[-r, r]$, hence integrable. We have

$$\int_{-r}^r \sqrt{r^2 - x^2}\,dx = r\int_{-r}^r \sqrt{1 - \left(\frac{x}{r}\right)^2}\,dx \overset{y:=\frac{x}{r},\ x=ry=:\psi(y)}{=} r\int_{-1}^1 \sqrt{1 - y^2}r\,dy = r^2\int_{-1}^1 \sqrt{1 - y^2}\,dy.$$

Setting $y = \sin\theta =: \psi(\theta)$, $\psi$ is a change of variable $\mathscr{C}^1$ and increasing between $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $[-1, 1]$. Therefore

$$\int_{-1}^{1} \sqrt{1-y^2}\, dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-(\sin\theta)^2}\cos\theta\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos\theta|\cos\theta\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^2\, d\theta.$$

Integrating by parts

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^2\, d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)(\sin\theta)'\, d\theta = [(\cos\theta)(\sin\theta)]_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-\sin\theta)(\sin\theta)\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin\theta)^2\, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1-(\cos\theta)^2\, d\theta = \pi - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^2\, d\theta\end{aligned}$$

whence

$$2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^2\, d\theta = \pi, \iff \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^2\, d\theta = \frac{\pi}{2}.$$

In conclusion, $\mathrm{Area}(x^2+y^2 \leqslant r^2) = 2\cdot\frac{\pi}{2}r^2 = \pi r^2$. $\square$

Example 8.4.4. *Compute*

$$\int_0^{\log 4} \frac{\sqrt{e^x-1}}{8e^{-x}+1}\, dx.$$

Sol. — Let $f(x) := \frac{\sqrt{e^x-1}}{8e^{-x}+1}$. Clearly $f \in \mathscr{C}([0,+\infty[)$ hence $f \in \mathscr{C}([0,\log 4]) \subset \mathscr{R}([0,\log 4])$. To compute the integral, let us apply the fundamental formula. We start computing a primitive for $f$:

$$\int \frac{\sqrt{e^x-1}}{8e^{-x}+1}dx\Big|_{y=\sqrt{e^x-1},\, x=\log(1+y^2)} = \int \frac{y}{8\frac{1}{1+y^2}+1}\frac{2y}{1+y^2}dy = \int \frac{2y^2}{y^2+9}dy = 2\int \frac{y^2+9-9}{y^2+9}dy$$

$$= 2\left[\int dy - 9\int \frac{1}{y^2+9}dy\right] = 2\left[y - \int \frac{1}{\left(\frac{y}{3}\right)^2+1}dy\right] = 2\left[y - 3\arctan\frac{y}{3}\right] = 2\sqrt{e^x-1} - 6\arctan\frac{\sqrt{e^x-1}}{3}.$$

Hence, from the fundamental formula of integral calculus, we get

$$\int_0^{\log 4} \frac{\sqrt{e^x-1}}{8e^{-x}+1}dx = 2\sqrt{e^x-1} - 6\arctan\frac{\sqrt{e^x-1}}{3}\Bigg|_{x=0}^{x=\log 4} = 2\sqrt{3} - \pi. \quad \square$$

## 8.5. Functions of integral type

Let us consider now a function of type

$$\Phi(x) := \int_{\alpha(x)}^{\beta(x)} f(y)\, dy.$$

Notice that $\alpha(x) \equiv c$ and $\beta(x) = x$ corresponds to $\Phi(x) \equiv F_c(x)$. In general, $\Phi$ is an extension of the concept of integral function, and for this reason it is called *function of integral type*. This kind of functions are used in several contexts and it is important to have the tools of calculus for them.

**Proposition 8.5.1**

Let $f \in \mathscr{C}([a,b])$, $\alpha, \beta : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be such that $\alpha(x), \beta(x) \in [a,b]$ for every $x \in I$. Suppose furthermore that $\alpha$ and $\beta$ be derivable on $I$. Then

$$\Phi(x) := \int_{\alpha(x)}^{\beta(x)} f(y)\, dy,\ x \in I,$$

is derivable and

$$\Phi'(x) = f(\beta(x))\beta'(x) - f(\alpha(x))\alpha'(x),\ x \in I.$$

Proof. Let $c \in [a,b]$ be fixed. Then

$$\Phi(x) = \int_{\alpha(x)}^{\beta(x)} f(y)\, dy = \int_{\alpha(x)}^{c} f(y)\, dy + \int_{c}^{\beta(x)} f(y)\, dy = -\int_{c}^{\alpha(x)} f(y)\, dy + \int_{c}^{\beta(x)} f(y)\, dy = -F_c(\alpha(x)) + F_c(\beta(x)).$$

Being $f$ continuous, by the fundamental theorem of calculus and chain rule

$$\Phi'(x) = -F_c'(\alpha(x))\alpha'(x) + F_c'(\beta(x))\beta'(x) = -f(\alpha(x))\alpha'(x) + f(\beta(x))\beta'(x). \quad \square$$

Example 8.5.2. *Compute*

$$\lim_{x \to 0} \frac{1}{x} \int_{x}^{2x} \frac{\sin y}{y}\, dy.$$

Sol. — We can think to $f(y) := \frac{\sin y}{y}$ as defined and continuous on $\mathbb{R}$ setting $f(0) = 1$. Clearly

$$\int_{x}^{2x} \frac{\sin y}{y}\, dy = \int_{x}^{0} \frac{\sin y}{y}\, dy + \int_{0}^{2x} \frac{\sin y}{y}\, dy \longrightarrow 0,$$

as $x \longrightarrow 0$, whence the limit is an indeterminate form $\frac{0}{0}$. By the Hôpital rule

$$\lim_{x \to 0} \frac{\int_x^{2x} \frac{\sin y}{y}\, dy}{x} \overset{(H)}{=} \lim_{x \to 0} \frac{\frac{\sin(2x)}{2x} \cdot 2 - \frac{\sin x}{x} \cdot 1}{1} = \lim_{x \to 0} \frac{\sin(2x) - \sin x}{x} \overset{(H)}{=} \lim_{x \to 0} \frac{2\cos(2x) - \cos x}{1} = 1. \quad \square$$

## 8.6. Generalized integrals

For many applications, the restrictions of the Riemann integral are too strong. In particular, the definition demands *bounded* functions on *closed and bounded* intervals $[a,b]$. This excludes two important cases:

- the case when the integration interval is unbounded as, for instance, for $\int_0^{+\infty} e^{-x}\, dx$;
- the case when the function is unbounded as, for instance, for $\int_0^1 \frac{1}{\sqrt{x}}\, dx$.

Let us start with the problem of giving a sense to $\int_a^{+\infty} f(x)\, dx$. It is natural to consider *partial area*

$$\int_a^R f(x)\, dx$$

and send $R \longrightarrow +\infty$. If the limit exists, we will say that the integral $\int_a^{+\infty} f(x)\, dx$ exists.

**Definition 8.6.1**

Let $a \in \mathbb{R}$, and consider a function $f : [a, +\infty[ \to \mathbb{R}$ such that $f \in \mathscr{R}([a, b])$ for every $b > a$. We set

$$\int_a^{+\infty} f(x)\, dx := \lim_{b \to +\infty} \int_a^b f(x)\, dx$$

provided the limit exists finite. In this case, we say that $f$ is **integrable in generalized sense on** $[a, +\infty[$.

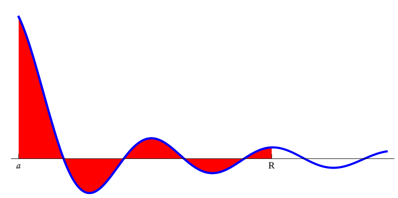
Similarly, let $b \in \mathbb{R}$, and consider a function $f : ]-\infty, b] \to \mathbb{R}$ such that $f \in \mathscr{R}([a, b])$ for every $a < b$. We set

$$\int_{-\infty}^b f(x)\, dx := \lim_{a \to -\infty} \int_a^b f(x)\, dx$$

provided the limit exists finite. In this case, we say that $f$ is **integrable in generalized sense on** $]-\infty, b]$.

Remark 8.6.2. If $f$ is integrable in generalized sense on $[a, +\infty[$ [resp. $f$ is integrable in generalized sense on $]-\infty, b]$] then, for every $\bar{a} > a$ for every $b < \bar{b}$, $f$ is integrable in generalized sense on $[a, +\infty[$ [resp. $f$ is integrable in generalized sense on $]-\infty, \bar{b}]$].

Condition $f \in \mathscr{R}([a, b])$ for every $b > a$ is for instance ensured by $f \in \mathscr{C}([a, +\infty[)$. Indeed, in this case $f \in \mathscr{C}([a, b])$ for every $b > a$, then $f \in \mathscr{R}([a, b])$. Of course, an analogous definition holds for $\int_{-\infty}^b f(x)\, dx$. Let us see some examples.



Example 8.6.3. *For any $a \in ]0, \infty[$ one has*

$$\exists \int_a^{+\infty} \frac{1}{x^\alpha}\, dx, \ (a > 0), \iff \alpha > 1.$$

Sol. — Of course $f(x) := \frac{1}{x^\alpha} \in \mathscr{C}(]0, +\infty[)$ so $f \in \mathscr{C}([a, +\infty[)$ for any $a > 0$. If $b > a$,

$$\int_a^b \frac{1}{x^\alpha}\, dx = \int_a^b x^{-\alpha}\, dx = \begin{cases} \left.\frac{x^{-\alpha+1}}{-\alpha+1}\right|_{x=a}^{x=b} = \frac{b^{-\alpha+1}}{-\alpha+1} - \frac{a^{-\alpha+1}}{-\alpha+1}, & \alpha \neq 1, \\ \\ \log x\Big|_{x=a}^{x=b} = \log b - \log a, & \alpha = 1. \end{cases}$$

By this is evident that

$$\lim_{b \to +\infty} \int_a^b \frac{1}{x^\alpha}\, dx = \begin{cases} \frac{a^{1-\alpha}}{\alpha - 1}, & \alpha > 1, \\ \\ +\infty, & \alpha \leqslant 1. \end{cases} \qquad \square$$

Similarly, for any $b < 0$, it is

$$\int_{-\infty}^{b} \frac{1}{(-x)^{\alpha}}\, dx < +\infty \iff \alpha > 1$$

(prove it by exercise).

EXAMPLE 8.6.4. *For any $a \in \mathbb{R}$,*

$$\exists \int_{a}^{+\infty} e^{\beta x}\, dx, \iff \beta < 0.$$

SOL. — Here $f(x) := e^{\beta x}$ is continuous on $\mathbb{R}$, therefore on $[a, +\infty[$. Moreover

$$\int_{a}^{R} e^{\beta x}\, dx = \begin{cases} \frac{e^{\beta x}}{\beta}\Big|_{x=a}^{x=R} = \frac{e^{\beta R}}{\beta} - \frac{e^{\beta a}}{\beta}, & \beta \neq 0, \\ \\ R - a, & \beta = 0. \end{cases}$$

It is now clear that $\lim_{b \to +\infty} \int_0^b e^{\beta x}\, dx \in \mathbb{R}$ iff $\beta < 0$. In such case

$$\int_{a}^{+\infty} e^{\beta x}\, dx = \lim_{b \to +\infty} \left( \frac{e^{\beta b}}{\beta} - \frac{e^{\beta a}}{\beta} \right) = -\frac{e^{\beta a}}{\beta}. \qquad \square$$

Similarly $\int_{-\infty}^{b} e^{\beta x}\, dx < +\infty$ iff $\beta > 0$ (exercise).

**Definition 8.6.5**

Consider a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in \mathscr{R}([a, b])$ for every $[a, b] \subset \mathbb{R}$. We set

$$\int_{-\infty}^{+\infty} f(x)\, dx := \int_{-\infty}^{c} f(x)\, dx + \int_{c}^{+\infty} f(x)\, dx,$$

provided both generalized integrals exist (it is easy to check that this does not depend on a specific $c$).

For instance,

$$\int_{\infty}^{+\infty} \frac{1}{1+x^2} = \int_{-\infty}^{0} \frac{1}{1+x^2} + \int_{0}^{+\infty} \frac{1}{1+x^2} = \lim_{a \to -\infty} \int_{a}^{0} \frac{1}{1+x^2} + \lim_{b \to +\infty} \int_{0}^{b} \frac{1}{1+x^2} =$$

$$\lim_{a \to -\infty} (\arctan 0 - \arctan a) + \lim_{b \to +\infty} \arctan b - \arctan 0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

REMARK 8.6.6. Notice that in general

$$\int_{-\infty}^{+\infty} f(x)\, dx \neq \lim_{a \to +\infty} \int_{-a}^{a} f(x)\, dx$$

Actually, the limit on the right-end side might exist even in the case when the function is not integrable in generalized sense. As an example, consider the function $f(x) := \dfrac{\operatorname{sgn}(x)}{1+|x|}$. One can easily see that it is *not* integrable in generalized sense both on $]-\infty,0]$ and on $[0,+\infty[$, so it is not integrable in generalized sense on $\mathbb{R}$. However, since $f$ is odd, one has

$$\int_{-a}^{a} f(x)dx = 0 \ \ \forall a \geq 0, \ \ \text{so that} \ \ \lim_{a\to+\infty}\int_{-a}^{a} f(x)\,dx = 0.$$

Let us consider now the case of a function $f:]a,b] \longrightarrow \mathbb{R}$ not defined at $x = a$ or of a function $f:[a,b[\longrightarrow \mathbb{R}$ not defined at $x = b$ (because of several reasons: for example, $f$ could be unbounded at $x$ tends to $a$ or to $b$).

**Definition 8.6.7**

Let $f \in \mathscr{R}(]\alpha,b])$, for every $a < \alpha < b$. We set

$$\int_{a}^{b} f(x)\,dx := \lim_{r\to a+}\int_{r}^{b} f(x)\,dx$$

provided the limit exists finite. In this case, we say that $f$ is **integrable in generalized sense on** $[a,b]$.
Similarly, let $f \in \mathscr{R}([a,\beta])$, for every $a < \beta < b$. We set

$$\int_{a}^{b} f(x)\,dx := \lim_{r\to a+}\int_{r}^{b} f(x)\,dx$$

provided the limit exists finite. In this case, we say that $f$ is **integrable in generalized sense on** $[a,b]$.

In the previous definition, we used the same notation of the Riemann integral, namely $\int_a^b f(x)\,dx$, to define also the generalized integral. This could potentially lead to some confusion if both are defined. There is no confusion at all: it is possible to prove that

**Proposition 8.6.8**

If $f \in \mathscr{R}([a,b])$ then $f$ is integrable in generalized sense on $[a,b]$ and the generalized integral coincides with the Riemann integral.

Example 8.6.9.

$$\exists \int_{a}^{b} \frac{1}{(x-a)^{\alpha}}\,dx, \iff \alpha < 1.$$

Sol. — Indeed $f(x) := \frac{1}{(x-a)^{\alpha}} \in \mathscr{C}(]a,b])$. Moreover

$$\int_{r}^{b} \frac{1}{(x-a)^{\alpha}}\,dx = \begin{cases} \frac{(x-a)^{-\alpha+1}}{-\alpha+1}\Big|_{x=r}^{x=b} = \frac{(b-a)^{-\alpha+1}}{-\alpha+1} - \frac{(r-a)^{-\alpha+1}}{-\alpha+1}, & \alpha \neq 1, \\ \\ \log(x-a)\Big|_{x=r}^{x=b} = \log(b-a) - \log(r-a), & \alpha = 1. \end{cases}$$

By this it is immediate to deduce that

$$\lim_{r\to a+}\int_r^b \frac{1}{(x-a)^\alpha}\,dx = \begin{cases} \frac{(b-a)^{-\alpha+1}}{-\alpha+1}, & \alpha<1,\\ \\ +\infty, & \alpha\geqslant 1. \end{cases} \qquad \square$$

A similar definition holds for $\int_a^b f(x)\,dx$ with $f:[a,b[\longrightarrow\mathbb{R}$.

Finally, if $f:]a,b[\longrightarrow\mathbb{R}$, we set

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

provided both generalized integrals $\int_a^c$ and $\int_c^b$ exist (this does not depend on $c\in]a,b[$). Finally, we may have cases when $f:]a,+\infty[\longrightarrow\mathbb{R}$. The idea is always the same:

$$\int_a^{+\infty} f(x)dx\colon = \int_a^c f(x)dx + \int_c^{+\infty} f(x)dx,$$

provided the two generalized integrals on the right-hand side do exist ( and definition turns out to be independent of $c\in]a,+\infty[$). An akin definition is given for functions defined on domain of the form $]-\infty,b[$ or even $]-\infty,\infty[$.

## 8.7. Convergence criteria for generalized integrals

**8.7.1. Constant sign integrand.** As it will happen for the series (see the next chapter), we start with the case of a constant sign integrand, for example $f\geqslant 0$. The first important result is the comparison test, at all similar to the same result for series:

**Theorem 8.7.1: comparison test**

Let $f,g\in\mathscr{R}([a,b])$ for every $b>a$, be such that

$$0\leqslant f(x)\leqslant g(x),\ \forall x\in[a,+\infty[.$$

Then

$$\exists\int_a^{+\infty} g(x)\,dx \implies \exists\int_a^{+\infty} f(x)\,dx.$$

We call $g$ an **integrable dominant** for $f$ on $[a,+\infty[$.

Proof. First, since $f\geqslant 0$, $\int_a^b f$ is increasing in $b$, therefore

$$\exists\lim_{b\to+\infty}\int_a^b f(x)dx \in\mathbb{R}\cup\{+\infty\}.$$

In short, we have to exclude the value $+\infty$. By monotonicity,

$$\int_a^b f(x)dx\leqslant\int_a^b g(x)dx$$

thus

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx \leqslant \lim_{b\to+\infty}\int_a^b g(x)dx = \int_a^{+\infty} g(x)dx < +\infty. \quad \square$$

Remark 8.7.2. it is trivial to verify that the comparison theorem holds true even id we weaken the hypothesis as follows: there exists $\eta \geq a$ such that

$$0 \leqslant f(x) \leqslant g(x), \ \forall x \in [\eta, +\infty[.$$

*Obviously, perfectly akin results hold true for the case when the function $f$ is defined on an interval of the form* $]-\infty, b]$

Example 8.7.3. *Discuss convergence for the Gauss integral*

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx$$

Sol. — We study convergence of $\int_0^{+\infty} e^{-x^2}\,dx$, since, provided the latter exists then $\int_{-\infty}^0 e^{-x^2}\,dx = \int_0^{+\infty} e^{-x^2}\,dx$ and

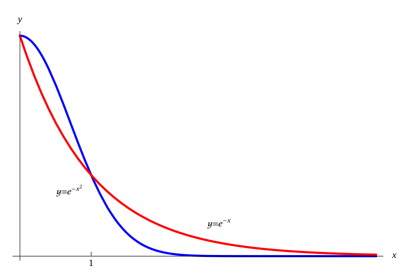
$$\int_{-\infty}^{+\infty} e^{-x^2} := \int_{-\infty}^{0} e^{-x^2}\,dx = \int_0^{+\infty} e^{-x^2}\,dx.$$

Clearly, $f(x) := e^{-x^2} \in \mathscr{C}([0,+\infty[)$ and $f \geqslant 0$. We want to compare $e^{-x^2}$ with $e^{-x}$. Notice that

$$e^{-x^2} \leqslant e^{-x}, \iff e^{x^2-x} \geqslant 1, \iff x^2 - x \geqslant 0.$$

content...

Since we are considering $x \geqslant 0$, this holds iff $x \geqslant 1$. Thus, on $[1,+\infty[$, $e^{-x}$ is an integrable dominant of $e^{-x^2}$. We conclude $\int_1^{+\infty} e^{-x^2}\,dx < +\infty$ and because of course $\int_0^1 e^{-x^2}\,dx$ exists (Riemann integral), we conclude that $\int_0^{+\infty} e^{-x^2}\,dx$ exists. $\square$




A fast way to check that a certain $g$ is an integrable dominant for $f$ at least on some neighbourhood of $+\infty$ is to check that

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)} = 0.$$

Indeed, for a suitable $R$, this leads to

$$\frac{f(x)}{g(x)} \leqslant 1, \ \forall x \geqslant R.$$

A stronger result is the

**Corollary 8.7.4: asymptotic comparison test**

Let $f, g \in \mathscr{R}([a,b])$ for every $b > a$, $f, g > 0$ on $[a, +\infty[$. Then, if $f \sim_{+\infty} g$ we have

$$\exists \int_a^{+\infty} f \iff \exists \int_a^{+\infty} g.$$

Example 8.7.5. *Discuss in function of $\alpha > 0$ the convergence of the integral*

$$\int_1^{+\infty} (x-1) \arctan \frac{1}{x^\alpha}\, dx.$$

Sol. — The function $f_\alpha(x) := (x-1)\arctan \frac{1}{x^\alpha} \in \mathscr{C}([1,+\infty[)$. Clearly, $f \geqslant 0$ on $[1,+\infty[$. Let us see the asymptotic behavior at $+\infty$. Being $\alpha > 0$, $\frac{1}{x^\alpha} \longrightarrow 0+$ as $x \longrightarrow +\infty$ and because $\arctan t \sim_0 t$, we have

$$f_\alpha(x) \sim_{+\infty} (x-1)\frac{1}{x^\alpha} \sim_{+\infty} \frac{x}{x^\alpha} = \frac{1}{x^{\alpha-1}},$$

therefore, by asymptotic comparison, $\exists \int^{+\infty} f_\alpha$ iff $\exists \int^{+\infty} \frac{1}{x^{\alpha-1}}\, dx$ iff $\alpha - 1 > 1$ that is iff $\alpha > 2$. □

Remark 8.7.6 (useful!). It is often impossible and useless to know exactly the sign of a function. When we apply the asymptotic comparison, we can say that if $f \sim_{+\infty} g$ and $g > 0$ then $f > 0$ in a neighborhood of $+\infty$. □

Here are the versions of the previous tests for generalized integrals of unbounded functions.

**Theorem 8.7.7: comparison**

Let $f, g \in \mathscr{R}(]r,b])$ for every $a < r < b$, be such that $0 \leqslant f(x) \leqslant g(x)$, $\forall x \in ]a,b]$. Then,

$$\exists \int_a^b g, \implies \exists \int_a^b f.$$

If moreover $f \sim_a g$, then

$$\exists \int_a^b g, \iff \exists \int_a^b f.$$

**8.7.2. Non constant sign integrand.** As it will be seen for the series (see next chapter), when we want to sum a generic function, things are much harder. As in that case, an important concept is

**Definition 8.7.8**

Let $f \in \mathscr{R}([a,b])$, for every $b > a$. We say that $f$ is **absolutely integrable on** $[a, +\infty[$ if

$$\exists \int_a^{+\infty} |f(x)|\, dx.$$

Analogous definition for all other generalized integrals. As for series

**Proposition 8.7.9**

An absolutely integrable function is integrable in generalized sense.

However, it is possible to show that *absolute integrability* is stronger than *integrability*. For instance, it is possible to prove that

$$\int_0^{+\infty} \frac{\sin x}{x}\, dx \in \mathbb{R}, \quad \text{but} \quad \int_0^{+\infty} \left|\frac{\sin x}{x}\right| dx = +\infty.$$

Example 8.7.10. *Discuss convergence for*

$$\int_0^{+\infty} \frac{\cos x}{x^2+1} dx.$$

Sol. — The integrand is continuous on $[0, +\infty[$, hence is locally integrable. Moreover

$$\left|\frac{\cos x}{x^2+1}\right| \leqslant \frac{1}{x^2+1} =: g(y), \quad \text{and} \quad \int_0^{+\infty} g(x)\, dx = \arctan x|_{x=0}^{x=+\infty} = \frac{\pi}{2}.$$

Therefore $\int_0^{+\infty} |f(x)|\, dx < +\infty$ by comparison, that is, the proposed integral is absolutely convergent. □

## 8.8. Exercises

Exercise 8.8.1. Compute

1. $\displaystyle\int_0^1 \frac{1}{1+x^4}\, dx.$
2. $\displaystyle\int_0^{\pi/3} \frac{1}{(\cos x)^3}\, dx.$
3. $\displaystyle\int_1^3 \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}\, dx.$
4. $\displaystyle\int_e^{e^2} \frac{\log x}{x}\sqrt{(\log x)^2-1}\; e^{\sqrt{(\log x)^2-1}}\, dx.$
5. $\displaystyle\int_0^{\log 4} \frac{\sqrt{e^x-1}}{8e^{-x}+1}\, dx.$
6. $\displaystyle\int_1^4 \frac{\log x}{x\sqrt{x}+2x+\sqrt{x}}\, dx.$
7. $\displaystyle\int_{1/4}^1 \frac{\sqrt{x}-1}{(\sqrt{x}+1)(\sqrt{x}-2)}\, dx.$
8. $\displaystyle\int_{-\log 3}^{-\log 2} \frac{e^{2x}+e^x}{e^{2x}-4e^x+3}\, dx.$
9. $\displaystyle\int_{\log 2}^{2\log 2} e^x \arctan\frac{1}{\sqrt{e^x-1}}\, dx.$
10. $\displaystyle\int_1^2 \frac{\left(\sinh\sqrt{\log x}\right)^2}{x}\, dx.$

Exercise 8.8.2. Compute the area interior to the ellypse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

Exercise 8.8.3. Compute

1. $\displaystyle\int_2^{+\infty} \frac{1}{x(\log x)^3}\,dx$. 2. $\displaystyle\int_1^{+\infty} \frac{1}{(1+x)\sqrt{x}}\,dx$. 3. $\displaystyle\int_1^{+\infty} \frac{2x\log x}{(1+x^2)^2}\,dx$. 4. $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^2+3x+4}\,dx$.

5. $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{x}\sqrt{2x+1}}\,dx$. 6. $\displaystyle\int_0^{2} \frac{1}{\sqrt{x(2-x)}}\,dx$. 7. $\displaystyle\int_0^{1} \frac{3x^2+2}{x^{3/2}}\,dx$. 8. $\displaystyle\int_0^{3} \frac{1}{\sqrt{3+2x-x^2}}\,dx$.

9. $\displaystyle\int_0^{1/2} \frac{1}{x(\log x)^2}\,dx$. 10. $\displaystyle\int_{-\infty}^{0} \frac{e^{2x}+e^x}{e^{2x}-2e^x+2}\,dx$.

Exercise 8.8.4 (★). Compute

1. $\displaystyle\lim_{x\to 0} \frac{x}{1-e^{x^2}} \int_0^x e^{y^2}\,dy$. 2. $\displaystyle\lim_{x\to 0} \frac{1}{x^2}\left(\int_0^{x^2} e^y\sqrt{y}\cos\sqrt{y}\,dy - x\cos x + x\right)$.

3. $\displaystyle\lim_{x\to 0} \frac{\int_0^x \left(e^{t^2}-7\sin(t^2)\right)\,dt - 12\sin x + 11x}{x^7}$.

Exercise 8.8.5 (★). Let

$$f(x) := \int_0^x \frac{\cos t}{1+t}\,dt, \quad g(x) := \int_0^{\sin x} \frac{1}{2+e^t}\,dt,$$

Say if $\frac{f}{g}$ is continuously extendable at $x=0$. In such a case, is the extension differentiable?

Exercise 8.8.6 (★★). Find $a, b, c \in \mathbb{R}$ such that

$$\int_x^{2x} \frac{\log t}{1+t}\,dt \sim_{+\infty} cx^a(\log x)^b.$$

Exercise 8.8.7 (★★). Find $C \neq 0$ and $\alpha \in \mathbb{R}$ such that

$$\int_x^{+\infty} e^{-y^2}\,dy \sim_{+\infty} C\frac{e^{-x^2}}{x^\alpha}.$$

Exercise 8.8.8. Compute

$$\int_0^1 (x-1)\arctan\frac{1}{x}\,dx.$$

Compute, if it exists $\int_0^1 f(x)\,dx$. Without computing any integral, determine the set of all $\alpha > 0$ such that $\int_1^{+\infty} f_\alpha(x)\,dx$ is finite. Are there values $\alpha > 0$ such that $\int_0^{+\infty} f_\alpha(x)\,dx$ converges?

Exercise 8.8.9. Let

$$f_\alpha(x) = x^\alpha \log\left(1+\frac{1}{x}\right).$$

Discuss integrability at $+\infty$ for $f_\alpha$ and compute, if it exists, $\int_1^{+\infty} f_{-\frac{1}{2}}(x)\,dx$.

Exercise 8.8.10. Determine $\alpha \in \mathbb{R}$ such that

$$\int_0^1 \frac{(\arctan x)^{\alpha+\frac{1}{2}}}{(1-\sqrt{x})^\alpha}\,dx,$$

converges and compute its value for $\alpha = -\frac{1}{2}$.

EXERCISE 8.8.11. Determine $\alpha, \beta \in \mathbb{R}$ such that the following integral converges

$$\int_0^{\frac{1}{2}} \frac{\arctan(x^\alpha)}{(1-\cos x)^\beta}\, dx$$

EXERCISE 8.8.12. Let

$$f_\alpha(x) := e^x \arctan \frac{1}{(e^x-1)^\alpha},\ x \in\in ]0, +\infty[.$$

Determine values $\alpha > 0$ such that $f_\alpha$ is integrable at 0+, at $+\infty$ and the integral $\int_0^{+\infty} f_\alpha(x)\, dx$ exists. Compute, if it exists, $\int_0^1 f_{1/2}(x)\, dx$.

EXERCISE 8.8.13. Let

$$f_\alpha(x) := \frac{1}{x^\alpha} \log(1+\sqrt[3]{x}),\quad x \in [0, +\infty[.$$

Compute, if it exists $\int_0^1 f_1(x)\, dx$. Hence, find all possible values of $\alpha$ such that $f$ is integrable at $+\infty$ ($\star$).

EXERCISE 8.8.14 ($\star$). Show that

$$\int_0^x \frac{\sin t}{t}\, dt \leqslant 1 + \log x,\quad \forall x \geqslant 1.$$

EXERCISE 8.8.15 ($\star\star$). Show that

$$\int_0^x \frac{\sin t}{1+t^2}\, dt \leqslant \frac{x^2}{2},\ \forall x \in \mathbb{R}.$$

EXERCISE 8.8.16 ($\star$). Let

$$\Phi(x) := \int_0^x \log\left(1 - e^{\frac{y-1}{y^2}}\right)\, dy.$$

Determine: the domain of $\Phi$, sign, limits (if finite or less) and asymptotes, continuity, differentiability, limits of $F'$, monotonicity, min/max, and plot a graph of $\Phi$.

EXERCISE 8.8.17 ($\star$). Let

$$\Phi(x) := \log\frac{4}{3} + \int_0^x \log\left(1 + \frac{t-1}{t^2+4}\right)\, dt.$$

Determine: the domain of $\Phi$, limits (if finite or less) and asymptotes, continuity, differentiability, limits of $F'$, monotonicity, min/max, convexity and plot a graph of $\Phi$. Deduce the sign of $\Phi$. Extra: compute $\Phi$ explicitly.

EXERCISE 8.8.18 ($\star$). Let

$$\Phi(x) := \int_0^x \frac{\sin y}{1+y^2}\, dy.$$

Determine: the domain of $\Phi$, symmetries, sign, limits (if finite or less) and asymptotes, continuity, differentiability, limits of $F'$, monotonicity, min/max, and plot a graph of $\Phi$. Find values $\alpha \in \mathbb{R}$ such that

$$\lim_{x\to 0+} \frac{f(x)}{x^\alpha} \in \mathbb{R}\backslash\{0\}.$$

CHAPTER 9

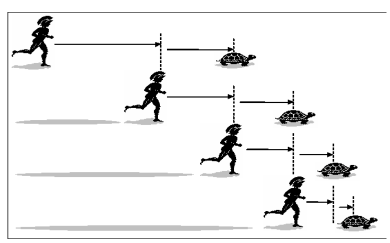
# Numerical Series

## 9.1. What is an Infinite Sum

One of the most famous problems of ancient logic is the famous *Zeno's paradox* on *impossibility of motion*. Imagine two runners, say a crippled Achilles and a Formula One Turtle. The track is 1km long and to give an advantage to the turtle, this starts 500m ahead with respect to Achilles. Of course, even if the crippled Achilles will run faster than the speedy Turtle, to simplify the calculations, we will assume that Achilles speed is $1km/h$ while Turtle run at $0,5km/h$. Intuition suggests that they reach together the finish line after $1h$. However, Zeno offered a different argument to prove that

*Achilles will never reach the Turtle!*

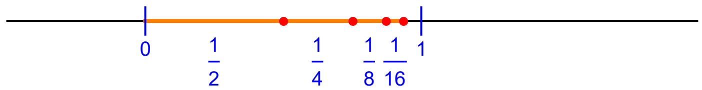
What is Zeno's argument? First, to reach the Turtle, Achilles must reach Turtle's initial position. When this happens, the Turtle will be a bit ahead to finish. Next, Achilles will have to reach this new Turtle's position, but when this happens the Turtle will be a bit ahead. Repeating this argument, we may argue that Achilles *never* reaches the turtle being *always* behind her. Where is the trouble with this argument?



The trouble is with meaning of *never*. Indeed, let us compute how long Achilles stays behind the turtle following Zeno's argument. To reach the first Turtle's position, Achilles has to run for 500m, so it takes $1/2h$. In this time, the Turtle runs $250m$ ahead. To cover these $250m$, Achilles takes $1/4h$. In this time, Turtle runs $125m$ ahead. To cover these $125m$ Achilles takes $1/8h$, in the same time Turtle will go $62,5m$ ahead. And so on. In practice, to compute the total time Achilles remains behind the Turtle, we have to compute the infinite sum

$$\frac{1}{2}h + \frac{1}{4}h + \frac{1}{8}h + \frac{1}{16}h + \frac{1}{32}h + \dots$$

The point is: is this sum finite? According to Zeno, the answer is yes. However, a simple figure suggests that the correct answer is no, and even that the sum equals $1h$.




We can formalize this intuition:

$$
\begin{aligned}
&a_0 = \tfrac{1}{2},\\
&a_0 + a_1 = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4},\\
&a_0 + a_1 + a_2 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{7}{8},\\
&a_0 + a_1 + a_2 + a_3 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} = \tfrac{15}{16},
\end{aligned}
$$

by which we may guess (we will prove precisely below)

$$a_0 + \cdots + a_n = \frac{1}{2} + \cdots + \frac{1}{2^{n+1}} = \frac{2^{n+1}-1}{2^{n+1}} = 1 - \frac{1}{2^{n+1}} \longrightarrow 1.$$

Thus

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{2^k} = 1.$$

The problem of computing the sum of an infinite number of numbers may be encountered in many other situations like exhaustion methods to compute the area of plane figures or in Probability Theory. In general, we aim to give a precise meaning to the operation

$$a_0 + a_1 + \ldots + a_k + a_{k+1} + \ldots \equiv \sum_{k=0}^{\infty} a_k, \quad \text{where } (a_k) \subset \mathbb{R}.$$

The point is that the sum is a binary operation and, thanks to the associative property, we may extend the operation of sum to the sum of any number of numbers. In particular, we can compute

$$s_n := a_0 + a_1 + \ldots + a_n \equiv \sum_{k=0}^{n} a_k.$$

We can do this calculation for every $n$ (at least in principle, in practice this is another story because each sum will cost a computation time, and we may imagine that if $n$ is big enough, the calculation of $s_n$ might be longer than the life of the universe. However, forgetting of the practical problem, let us say that any $s_n$ can be computed. The natural idea would be then to "take $n = +\infty$". The way we may do this is to let $n \longrightarrow +\infty$, that is, to compute

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \sum_{k=0}^{n} a_k.$$

If the limit exists, this could be taken as the value of the infinite sum. This turns out to be the right idea as we will see. Nonetheless, there is no doubt that series are a difficult guest and often students are confused.

The aim of this chapter is to introduce the concept and methods of infinite sums, mathematically called *series*. This actually does not seem a nice name because it sounds like a synonymous of *sequence*, leading someone to confuse *series* with *sequences*. The confusion is twofold because, as the last formula points out, we will use the limit of suitable sequences to define convergent sums! Since this terminology is universally accepted, we will not change it, so we advice the student to pay particular attention, especially when he/she begins to enter in this chapter.

## 9.2. Definition and examples

We start formalizing the main idea given in the Introduction:

**Definition 9.2.1**

Let $(a_n) \subset \mathbb{R}$ and define the sequence $(s_n)$ as

$$s_n := \sum_{k=0}^{n} a_k = a_0 + \cdots + a_n. \quad (n \text{ th partial sum})$$

We say that the series $\sum_{n=0}^{\infty} a_n$ is

- **convergent** if $\exists \lim_n s_n \in \mathbb{R}$.
- **divergent** if $\exists \lim_n s_n = \pm\infty$.
- **indeterminate** if $\nexists \lim_n s_n$.

For a convergent series, we call **sum of the series** the number

$$\sum_{n=0}^{\infty} a_n := \lim_n s_n \equiv \lim_n \sum_{k=0}^{n} a_k.$$

Example 9.2.2 (Geometric series). *Let $q \in \mathbb{R}$. Then*

$$\sum_{n=0}^{\infty} q^n \begin{cases} = \frac{1}{1-q} \in \mathbb{R} \ (\textit{converges}), & \textit{if } |q| < 1. \\ = +\infty, \ (\textit{diverges}), & \textit{if } q \geqslant 1, \\ \textit{indeterminate}, & \textit{if } q \leqslant -1. \end{cases} \tag{9.2.1}$$

Sol. — Let us compute the partial sums. Recalling the remarkable identity

$$1 - q^{N+1} = (1-q)(1 + q + q^2 + \cdots + q^N), \implies s_N = \sum_{n=0}^{N} q^n = \frac{1 - q^{N+1}}{1-q}, \text{ if } q \neq 1.$$

If $q = 1$

$$s_N = \sum_{n=0}^{N} 1^n = \sum_{n=0}^{N} 1 = N + 1.$$

Therefore

$$s_N = \sum_{n=0}^{N} q^n = \begin{cases} \dfrac{1 - q^{N+1}}{1-q}, & q \neq 1, \\ N + 1, & q = 1. \end{cases}$$

Clearly, if $q = 1$, $s_N = N + 1 \longrightarrow +\infty$, That is, the series is divergent. Let us study now the case $q \neq 1$. We have to compute

$$\lim_{N \to +\infty} s_N = \lim_{N \to +\infty} \frac{1 - q^{N+1}}{1-q} = \frac{1}{1-q} - \frac{1}{1-q} \lim_{N \to +\infty} q^{N+1}.$$

But $q^{N+1}$ is an exponential and

$$q^{N+1}\begin{cases} \longrightarrow 0, & |q|<1,\\ \longrightarrow +\infty, & q>1,\\ \text{doesn't have a limit}, & q\leqslant -1. \end{cases}$$

By this the conclusion easily follows. □

Example 9.2.3 (Mengoli series).

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1.$$

Sol. — Let's compute the $n$th partial sum:

$$\begin{aligned} s_n &= \textstyle\sum_{k=1}^{n}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= 1-\frac{1}{n+1}\longrightarrow 1. \quad \square \end{aligned}$$

These examples may induce the false idea that to study the convergence and sum of a series is relatively easy: first, we compute the partial sum $s_n$, hence its limit for $n\longrightarrow+\infty$. The problem is that we are not always able to compute $s_n$ in a useful form to compute its limit in $n$. Nonetheless, we may still be able to say if the series converges or less.

Example 9.2.4 (harmonic series).

$$\sum_{n=1}^{\infty}\frac{1}{n}=+\infty.$$

Sol. — By definition,

$$s_n=\sum_{k=1}^{n}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}.$$

To compute the limit we cannot use this form. Computing few sums as

$$s_{10}=2,92897 \quad s_{100}=5,18738 \quad s_{1.000}=7,48547 \quad s_{10.000}=9,78561 \quad s_{100.000}=12,0901\ \ldots$$

we see clearly that $s_n$ increases (evident: we are summing positive numbers), but the growth is not particularly fast. Since $s_n\nearrow$ we know (monotonic sequences) that either $s_n\longrightarrow s<+\infty$ (in this case the series converges) or $s_n\longrightarrow+\infty$ (divergence). Thus, for sure this series cannot be indeterminate. It remains to know if is it convergent or less. To respond to this question, we show now an ingenious method due to Cauchy: take $n$ equal to a power of 2, that is $n=2,4,8,16,32,64,\ldots$, in general $n=2^m$: then

$$\begin{aligned} s_{2^m} &= 1+\tfrac{1}{2}+\left(\tfrac{1}{3}+\tfrac{1}{4}\right)+\left(\tfrac{1}{5}+\cdots+\tfrac{1}{8}\right)+\left(\tfrac{1}{9}+\cdots+\tfrac{1}{16}\right)+\cdots+\left(\tfrac{1}{2^{m-1}+1}+\cdots+\tfrac{1}{2^m}\right)\\ &\geqslant 1+\tfrac{1}{2}+\left(\tfrac{1}{4}+\tfrac{1}{4}\right)+\left(\tfrac{1}{8}+\cdots+\tfrac{1}{8}\right)+\left(\tfrac{1}{16}+\cdots+\tfrac{1}{16}\right)+\cdots+\left(\tfrac{1}{2^m}+\cdots+\tfrac{1}{2^m}\right)\\ &= 1+\tfrac{1}{2}+2\tfrac{1}{4}+4\tfrac{1}{8}+8\tfrac{1}{16}+\cdots+2^{m-1}\tfrac{1}{2^m}\\ &= 1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\cdots+\tfrac{1}{2}=1+\tfrac{m}{2}, \end{aligned}$$

that is

$$s_{2^m} \geqslant 1 + \frac{m}{2}.$$

Now, let $n$ be such that $2^m \leqslant n < 2^{m+1}$. Then $m \leqslant \log_2 n < m+1$ and

$$s_n \geqslant s_{2^m} \geqslant 1 + \frac{m}{2} \geqslant 1 + \frac{\log_2 n - 1}{2} \longrightarrow +\infty, \; n \longrightarrow +\infty. \qquad \square$$

This example helps to drive out a (false) common belief: *a series* $\sum_n a_n$ *converges iff* $a_n \longrightarrow 0$. This is false, as Example 9.2.4 shows. However, the necessity is true:

**Proposition 9.2.5**

Let $(a_n) \subset \mathbb{R}$. If $\sum_n a_n$ converges then, necessarily, $a_n \longrightarrow 0$.

Proof. Very easy: let $s_n := \sum_{k=0}^n a_k$. Then, if $s_n \longrightarrow s \in \mathbb{R}$ (that is the series $\sum_n a_n$ converges) we have

$$a_n = s_n - s_{n-1} \longrightarrow s - s = 0. \qquad \square$$

As we pointed out with the harmonic series, the necessary condition is not sufficient. Next Sections are devoted to present a number of sufficient conditions that in certain conditions may be helpful to discuss convergence.

## 9.3. Constant sign terms series

A *constant sign series* is

$$\sum_{n=0}^{\infty} a_n, \quad \text{with } (a_n) \subset [0, +\infty[, \text{ or } (a_n) \subset ]-\infty, 0].$$

Since we may always change sign to all terms, hereafter we will assume $a_n \geqslant 0$ for every $n$. We use the notation $(a_n) \geqslant 0$. As we learnt from harmonic series,

**Proposition 9.3.1**

Every constant sign term series is either convergent or divergent.

Proof. If $a_n \geqslant 0$, $\forall n \in \mathbb{N}$, partial sums are increasing because

$$s_{n+1} = \sum_{k=0}^{n+1} a_k = \sum_{k=0}^{n} a_k + a_{n+1} = s_n + a_{n+1} \geqslant s_n.$$

By Thm 4.6, there exists $\lim_n s_n \in \mathbb{R} \cup \{+\infty\}$. $\square$

Thus, to show the convergence we need somehow to prove that $\sum_n a_n < +\infty$. A natural idea is *to compare* $\sum_n a_n$ with another known sum $\sum_n b_n$ where the $b_n$ are greater than the $a_n$:

**Theorem 9.3.2: comparison test**

Let $(a_n) \geqslant 0$ be such that

$$a_n \leqslant b_n, \text{ definitely (that is for } n \geqslant N).$$

Then

$$\sum_n b_n \text{ converges} \implies \sum_n a_n \text{ converges.}$$

We call $(b_n)$ a **convergent dominant** of $(a_n)$.

Proof. For simplicity, we will assume that

$$a_n \leqslant b_n, \ \forall n \in \mathbb{N}.$$

Then

$$s_n = \sum_{k=0}^{n} a_k \leqslant \sum_{k=0}^{n} b_k =: \widetilde{s}_n, \ \forall n.$$

Since $s_n, \widetilde{s_n} \nearrow$,

$$\lim_n s_n = \sup\{s_n \ : \ n \in \mathbb{N}\} \leqslant \sup\{\widetilde{s_n} \ : \ n \in \mathbb{N}\} = \lim_n \widetilde{s_n} < +\infty,$$

thus $\lim_n s_n < +\infty$ as well. □

Example 9.3.3.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \ \textit{converges.}$$

Sol. — This series presents the same difficulties of the harmonic series $\sum_n \frac{1}{n}$. However, we can easily compare it with a Mengoli–type series noticing that

$$\frac{1}{n^2} = \frac{1}{n \cdot n} \leqslant \frac{1}{n(n-1)}, \ \forall n \geqslant 2.$$

We have that $\frac{1}{n(n-1)}$ is a convergent dominant because

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

By comparison test we deduce that $\sum_n \frac{1}{n^2}$ converges. □

Example 9.3.4.

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \ \textit{converges} \ \forall \alpha \geqslant 2.$$

Sol. — Let $\alpha > 2$. Then

$$\frac{1}{n^\alpha} \leqslant \frac{1}{n^2}, \ \forall n \geqslant 1.$$

By previous example $\sum_n \frac{1}{n^2}$ converges, so $\sum_n \frac{1}{n^\alpha}$ converges by comparison. □

The comparison test may be applied in the contrary direction: if a series $\sum_n a_n$ dominates a divergent series $\sum_n b_n$, then $\sum_n a_n$ it cannot be convergent (otherwise the dominated would be convergent by comparison!).

Example 9.3.5.

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \textit{ diverges } \forall \alpha \leqslant 1.$$

Sol. — Let $\alpha < 1$. Notice that $n^\alpha \leqslant n$ for any $n \geqslant 1$, so

$$\frac{1}{n} \leqslant \frac{1}{n^\alpha}, \ \forall n \geqslant 1.$$

By this follows that $\sum_n \frac{1}{n^\alpha}$ dominates $\sum_n \frac{1}{n}$ which is divergent, and this forces to be divergent also its dominant. □

Notice that we proved

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}, \begin{cases} \text{converges if } \alpha \geqslant 2, \\ \\ \text{diverges if } \alpha \leqslant 1. \end{cases}$$

What can be said in the cases $1 < \alpha < 2$? Adapting Cauchy argument, we have

**Proposition 9.3.6**

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \text{ converges}, \iff \alpha > 1. \tag{9.3.1}$$

Proof. We already proved that the series diverges for $\alpha \leqslant 1$. Let $\alpha > 1$ and consider partial sum $s_{2^m-1}$:

$$\begin{aligned} s_{2^m} &= 1 + \left(\tfrac{1}{2^\alpha} + \tfrac{1}{3^\alpha}\right) + \left(\tfrac{1}{4^\alpha} + \cdots + \tfrac{1}{7^\alpha}\right) + \cdots + \left(\tfrac{1}{(2^{m-1})^\alpha} + \cdots + \tfrac{1}{(2^m-1)^\alpha}\right) \\ &\leqslant 1 + 2\tfrac{1}{2^\alpha} + 4\tfrac{1}{4^\alpha} + \cdots + 2^{m-1}\tfrac{1}{(2^{m-1})^\alpha} \\ &= \textstyle\sum_{k=0}^{m-1} \tfrac{1}{(2^k)^{\alpha-1}} = \sum_{k=0}^{m-1} \left(\tfrac{1}{2^{\alpha-1}}\right)^k \\ &\leqslant \textstyle\sum_{k=0}^{\infty} \left(\tfrac{1}{2^{\alpha-1}}\right)^k = \tfrac{1}{1-\frac{1}{2^{\alpha-1}}} = \tfrac{2^{\alpha-1}}{2^{\alpha-1}1}. \end{aligned}$$

Notice that we used the convergence of the geometric series of ratio $\frac{1}{2^{\alpha-1}} < 1$ just because $\alpha > 1$. A fortiori,

$$s_n \leqslant \frac{2^{\alpha-1}}{2^{\alpha-1}-1}, \ \forall n \in \mathbb{N}.$$

Thus, in particular, $(s_n)$ is upper bounded. Since $s_n \nearrow s$ we conclude $s < +\infty$, that is, the series converges. □

The difficulty applying the comparison test is in the search for a convergent dominant. It is a good idea to have a more versatile version of the comparison test:

**Corollary 9.3.7**

Let $(a_n) \geqslant 0$.

$$\text{If } a_n \ll_{+\infty} b_n, \text{ and } \sum_n b_n \text{ converges}, \implies \sum_n a_n \text{ converges}.$$

Proof. Indeed: $a_n \ll b_n$ means $\frac{a_n}{b_n} \longrightarrow 0$, therefore

$$0 \leqslant \frac{a_n}{b_n} \leqslant 1, \text{ definitely}, \iff 0 \leqslant a_n \leqslant b_n, \text{ definitely}.$$

By this the conclusion follows. □

Example 9.3.8. *Discuss convergence of*

$$\sum_{n=0}^{\infty} e^{-\sqrt{n}}.$$

Sol. — Of course, it is not a geometric series! Clearly, $(a_n) \geqslant 0$. The necessary condition $a_n \longrightarrow 0$ is fulfilled, but unfortunately this is not sufficient to ensure convergence. It is however natural to think that

$$e^{-\sqrt{n}} \ll_{+\infty} \frac{1}{n^2}.$$

Indeed

$$\frac{e^{-\sqrt{n}}}{\frac{1}{n^2}} = \frac{n^2}{e^{\sqrt{n}}} = e^{2\log n - \sqrt{n}} \longrightarrow 0, \text{ being } 2\log n - \sqrt{n} = -\sqrt{n}\left(1 - \frac{2\log n}{\sqrt{n}}\right) \longrightarrow -\infty$$

and $n^{1/2} \gg_{+\infty} \log n$. Therefore, the series is convergent. □

**9.3.1. Asymptotic comparison.** It seems intuitive that if $a_n \sim_{+\infty} b_n$ the sums $\sum_n a_n$ and $\sum_n b_n$ have the same behaviour:

**Corollary 9.3.9**

Let $(a_n), (b_n) \geqslant 0$ such that $a_n \sim_{+\infty} b_n$. Then

$$\sum_n a_n \text{ converges} \iff \sum_n b_n \text{ converges}.$$

Proof. We know that

$$\frac{a_n}{b_n} \longrightarrow 1.$$

Therefore there exists an $N$ such that

$$\frac{1}{2} \leqslant \frac{a_n}{b_n} \leqslant 2, \ \forall n \geqslant N, \iff \frac{1}{2}b_n \leqslant a_n \leqslant 2b_n, \ \forall n \geqslant N. \tag{9.3.2}$$

Then, $2b_n$ is a dominant for $a_n$, and $2a_n$ is a dominant for $b_n$. By this easily: if $\sum_n a_n$ converges, of course $\sum_n (2a_n)$ converges as well, hence $\sum_n b_n$ converges (by comparison) and vice versa. □

Remark 9.3.10. *If $a_n \sim_{+\infty} b_n$ and $a_n > 0$, then $b_n > 0$ definitely.* Indeed:

$$\frac{a_n}{b_n} \longrightarrow 1, \implies \frac{a_n}{b_n} \geqslant \frac{1}{2} > 0, \text{ definitively,}$$

and by this the conclusion follows. □

In particular:

- if $a_n \sim_{+\infty} \frac{C}{n^\alpha}$ for some $C \neq 0$ and $\alpha \in \mathbb{R}$, then $\sum_n a_n$ converges iff $\alpha > 1$.
- if $a_n \sim_{+\infty} Cq^n$ for some $C \neq 0$ and $q \geqslant 0$, then $\sum_n a_n$ converges iff $q < 1$.

Example 9.3.11. *Discuss the convergence of the series*

$$\sum_{n=0}^{\infty} \frac{n}{n^\alpha + 1}, \quad \alpha \in \mathbb{R}.$$

Sol. — Let $a_n := \frac{n}{n^\alpha+1}$. Clearly $(a_n) \geqslant 0$, so we have a constant sign term series. Moreover

$$a_n = \frac{n}{n^\alpha + 1} = \frac{n}{n^\alpha \cdot 1_n} = \frac{1_n}{n^{\alpha-1}} \sim \frac{1}{n^{\alpha-1}}.$$

Therefore: by asymptotic comparison $\sum_n a_n$ converges iff $\alpha - 1 > 1$ that is $\alpha > 2$. □

**9.3.2. Root and Ratio tests.** If $a_n \sim_{+\infty} Cq^n$ then, assuming for instance $C > 0$,

$$a_n^{1/n} \sim_{+\infty} C^{1/n} q \longrightarrow q, \quad \frac{a_{n+1}}{a_n} \sim_{+\infty} \frac{Cq^{n+1}}{Cq^n} = q \longrightarrow q,$$

so that

$$q = \lim_n a_n^{1/n} = \lim_n \frac{a_{n+1}}{a_n},$$

and according to $q < 1$ or $q \geqslant 1$ we could say if the series converges or not. This is essentially the content of the two following tests.

**Proposition 9.3.12: root test**

Let $(a_n) \geqslant 0$ and suppose

$$\exists \lim_{n \to +\infty} \sqrt[n]{a_n} \equiv \lim_{n \to +\infty} a_n^{1/n} =: q \ (\in \mathbb{R}_+ \cup \{+\infty\}).$$

Then

- if $q < 1$ the series $\sum_n a_n$ converges.
- if $q > 1$ (included $q = +\infty$) the series $\sum_n a_n$ diverges and $a_n \longrightarrow +\infty$.

If $q = 1$, nothing can be said based on the test (that is: the test fails).

Proof. Let's start with the case $q < 1$. By the definition of limit,

$$\forall \varepsilon > 0, \ \exists N, \ q - \varepsilon \leqslant a_n^{1/n} \leqslant q + \varepsilon, \ \forall n \geqslant N. \tag{9.3.3}$$

Let $\varepsilon > 0$ small enough in such a way that $q + \varepsilon < 1$ (this is possible because $q < 1$). Therefore, by (9.3.3)

$$a_n^{1/n} \leqslant q + \varepsilon, \text{ definitely}, \iff a_n \leqslant (q + \varepsilon)^n, \text{ definitely}.$$

Being $q+\varepsilon<1$, the series $\sum_n (q+\varepsilon)^n$ converges and by comparison it converges also $\sum_n a_n$.

Consider now the case $q>1$, $q\in\mathbb{R}$ (we leave $q=+\infty$ to the reader). The (9.3.3) is still true: but now we choose $\varepsilon>0$ such that $q-\varepsilon>1$ (this is actually possible because $q>1$). Then

$$a_n^{1/n} \geqslant q-\varepsilon, \text{ definitely}, \iff a_n \geqslant (q-\varepsilon)^n, \text{ definitely}.$$

Now, being $q-\varepsilon>1$, we know that $(q-\varepsilon)^n \longrightarrow +\infty$, therefore $a_n \longrightarrow +\infty$. This means that the necessary condition is not fulfilled and the proof is finished. □

Example 9.3.13. *Discuss convergence of*

$$\sum_{n=1}^{\infty} \frac{(n-1)^{n^2}}{n^{n^2}}.$$

Sol. — Clearly, $a_n = \frac{(n-1)^{n^2}}{n^{n^2}} > 0$, so we have a constant sign term series. Applying the root test:

$$a_n^{1/n} = \left(\frac{(n-1)^{n^2}}{n^{n^2}}\right)^{1/n} = \frac{(n-1)^n}{n^n} = \left(1-\frac{1}{n}\right)^n \longrightarrow e^{-1} < 1.$$

We conclude that the series is convergent. □

Remark 9.3.14. What does it mean "*the test fails*"? It means that there are convergent series for which $q=1$ and divergent series for which, again, $q=1$. In other words: $q=1$ does not distinguish between convergent and divergent series. For instance:

- the series $\sum_n \frac{1}{n}$ diverges and
$$a_n^{1/n} = \left(\frac{1}{n}\right)^{1/n} = \frac{1}{n^{1/n}} = n^{-\frac{1}{n}} = e^{-\frac{\log n}{n}} \longrightarrow 1;$$
- the series $\sum_n \frac{1}{n^2}$ converges and
$$a_n^{1/n} = \left(\frac{1}{n^2}\right)^{1/n} = \frac{1}{(n^{1/n})^2} = n^{-\frac{2}{n}} = e^{-2\frac{\log n}{n}} \longrightarrow 1. \quad \square$$

A twin test of root test (with similar proof) is the

**Proposition 9.3.15: ratio test**

Let $(a_n)>0$ and suppose that

$$\exists \lim_{n\to+\infty} \frac{a_{n+1}}{a_n} =: q \ (\in \mathbb{R}_+ \cup \{+\infty\}).$$

Then

- if $q<1$ the series $\sum_n a_n$ converges.
- if $q>1$ (included $q=+\infty$) the series $\sum_n a_n$ diverges and $a_n \longrightarrow +\infty$.

In the case $q=1$ nothing can be said based on the test (that is: the test fails).

Example 9.3.16. *Discuss convergence for*

$$\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}.$$

Sol. — Clearly $a_n := \frac{2^n n!}{n^n} > 0$ for any $n \geqslant 1$. Applying the ratio test:

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}} = 2\frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{n+1}} = 2(n+1)\left(\frac{n}{n+1}\right)^n \frac{1}{n+1} = 2\left(\frac{n}{n+1}\right)^n \\ &= \frac{2}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{2}{e} < 1,\end{aligned}$$

by which we conclude that the series converges. □

Remark 9.3.17. Basically, root and ratio tests are equivalent. It could be proved that if the ratio test give answer $q$ then also the root give answer $q$, but the contrary might be false. Here is an example: consider the series

$$\sum_{n=0}^{\infty} 2^{(-1)^n - n}.$$

Then

$$\frac{a_{n+1}}{a_n} = \frac{2^{(-1)^{n+1}-(n+1)}}{2^{(-1)^n - n}} = 2^{-2(-1)^n - 1} = \frac{1}{8},\ 2,\ \frac{1}{8},\ 2, \dots,$$

does not exist. On the other hand,

$$a_n^{1/n} = 2^{\frac{(-1)^n - n}{n}} = 2^{\frac{(-1)^n}{n} - 1} \longrightarrow 2^{-1} = \frac{1}{2}.$$

Therefore, the root test works correctly and we can deduce the convergence. □

## 9.4. Variable sign series

We pass now to the general case of a series $\sum_n a_n$ with $(a_n) \subset \mathbb{R}$. General discussion is much more complex and it might be very difficult to say if a series converges or less. We will limit here to a few important cases.

### 9.4.1. Alternate sign series.

We start with a case where the sign varies in a very "regular" way, in the sense that the signs are $+, -, +, -, \dots$. Formally, we consider a series of type

$$\sum_{n=0}^{\infty} (-1)^n b_n = b_0 - b_1 + b_2 - b_3 + b_4 - b_5 + \dots, \quad \text{where } (b_n) \geqslant 0.$$

These series are called **alternate sign series**. The main tool is the

**Theorem 9.4.1: Leibniz test**

Let $(b_n) \geqslant 0$ be such that

- $b_n \searrow$ definitely (that is $b_{n+1} \leqslant b_n$ definitely);
- $b_n \longrightarrow 0$.

Then $\sum_{n=0}^{\infty}(-1)^n b_n$ converges. Furthermore, if $s_n$ is the $n-$th partial sum and $s$ is the sum of the series, the following bound holds:

$$|s - s_n| \leqslant b_{n+1}. \tag{9.4.1}$$

Proof. The key point is that $s_{2n}$ decreases while $s_{2n+1}$ increases. Indeed

$$s_{2n+2} = s_{2n} - b_{2n+1} + b_{2n+2} = s_{2n} + (b_{2n+2} - b_{2n+1}) \leqslant s_{2n},$$

because $b_n \searrow$, hence $b_{2n+2} \leqslant b_{2n+1}$. Therefore $(s_{2n}) \searrow$, hence

$$\exists \lim s_{2n} =: s.$$

Similarly

$$s_{2n+3} = s_{2n+1} + (b_{2n+2} - b_{2n+3}) \geqslant s_{2n+1},$$

that is $(s_{2n+1}) \nearrow$, hence

$$\exists \lim s_{2n+1} =: \widetilde{s}.$$

Moreover, since

$$s_{2n+1} = s_{2n} - b_{2n+1}$$

letting $n \longrightarrow +\infty$, we have

$$\widetilde{s} = s.$$

From this easily follows that $s_n \longrightarrow s$. Let us come finally to the estimate: since

$$s_{2n} - b_{2n+1} = s_{2n+1} \leqslant s, \quad \Longrightarrow \quad 0 \leqslant s_{2n} - s \leqslant b_{2n+1},$$

and similarly

$$s_{2n+1} + b_{2n+2} = s_{2n+2} \geqslant s, \quad \Longrightarrow \quad 0 \leqslant s - s_{2n+1} \leqslant b_{2n+2}.$$

We can always write these two as $|s_n - s| \leqslant b_{n+1}$. □

Example 9.4.2. *Discuss convergence for the series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

*Determine an approximation of the sum by less than* $\frac{1}{100}$.

Sol. — We have $\sum_n (-1)^n b_n$ with $b_n := \frac{1}{n} \geqslant 0$. Clearly $b_n \searrow 0$, so the convergence follows by Leibniz test. To respond to the second question, recall the bound (9.4.1), that is in this case becomes

$$|s - s_n| \leqslant b_{n+1} = \frac{1}{n+1},$$

imposing $\frac{1}{n+1} < \frac{1}{100}$, that is $n > 99$, we have $|s - s_n| < \frac{1}{100}$. Thus, $s_{100}$ is an approximation of $s$ with an error $< \frac{1}{100}$. □

EXAMPLE 9.4.3. *Discuss convergence for the series*

$$\sum_{n=3}^{\infty}(-1)^n \frac{\log n}{(\log n)^2 - 1}.$$

SOL. — Let $b_n := \frac{\log n}{(\log n)^2-1}$, $n \geqslant 3$. Being $\log n > \log e = 1$ as $n \geqslant 3$, $(\log n)^2 - 1 > 0$ as $n \geqslant 3$, so $b_n > 0$. This means that we have an alternate sign term series. Let us apply the Leibniz test. First notice that

$$b_n = \frac{\log n}{(\log n)^2 - 1} = \frac{\log n}{(\log n)^2\left(1 - \frac{1}{(\log n)^2}\right)} = \frac{1}{\log n}\frac{1}{1_n} \longrightarrow 0.$$

Now we want to check if $b_n \searrow$. This is not evident. We have

$$\begin{aligned} b_{n+1} \leqslant b_n, \quad &\Longleftrightarrow \frac{\log(n+1)}{(\log(n+1))^2 - 1} \leqslant \frac{\log n}{(\log n)^2 - 1}, \\ &\Longleftrightarrow (\log(n+1))\left((\log n)^2 - 1\right) \leqslant (\log n)\left((\log(n+1))^2 - 1\right) \\ &\Longleftrightarrow (\log n)(\log(n+1))\left(\log(n+1) - \log n\right) \geqslant \log n - \log(n+1), \\ &\Longleftrightarrow (\log n)(\log(n+1)) \geqslant -1, \quad (\text{being } \log(n+1) - \log n > 0) \end{aligned}$$

This is evident for $n \geqslant 3$. Therefore, Leibniz test applies, and the series converges. □

**9.4.2. Absolute convergence.** In the general case of a series $\sum_n a_n$ with $(a_n) \subset \mathbb{R}$ with no particular informations on sign of $a_n$, the most important tool is the following

**Theorem 9.4.4**

Let $(a_n) \subset \mathbb{R}$. Then

$$\sum_{n=0}^{\infty} |a_n| \text{ converges} \implies \sum_{n=0}^{\infty} a_n \text{ converges.}$$

If $\sum_n |a_n|$ converges, we say that $\sum_n a_n$ is **absolutely convergent**.

PROOF. Define

$$a_n^+ := \max\{a_n, 0\}, \quad a_n^- := \max\{-a_n, 0\}.$$

Clearly $a_n^\pm \geqslant 0$, $a_n^+ + a_n^- = |a_n|$ while $a_n^+ - a_n^- = a_n$. In particular

$$0 \leqslant a_n^\pm \leqslant |a_n|.$$

By comparison then, $\sum_k a_k^\pm$ are both convergent hence, easily, also $\sum_k (a_k^+ - a_k^-) = \sum_k a_k$ converges. □

REMARK 9.4.5. *The sufficient condition is not necessary:* for instance, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is simply convergent by Leibniz test but not absolutely convergent because $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_n \frac{1}{n}$. □

Example 9.4.6. *Discuss in function of $x \in ]0, +\infty[$ simple and absolute convergence for*

$$\sum_{n=1}^{\infty} \frac{(\log x)^n}{\sqrt{n(n+1)}}.$$

Sol. — Let $a_n := \frac{(\arctan x)^n}{\sqrt{n(n+1)}}$. Let us start by absolute convergence. This means to study the series

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{|\log x|^n}{\sqrt{n(n+1)}}.$$

Applying the root test

$$|a_n|^{\frac{1}{n}} = \frac{|\log x|}{(n(n+1))^{\frac{1}{2n}}} = |\log x| \frac{1}{(n^{1/n})^2((n+1)^{1/n})^2} \longrightarrow |\log x|$$

being $n^{1/n} = e^{\frac{\log n}{n}} \longrightarrow 1$ and similarly $(n+1)^{1/n} = e^{\frac{\log(n+1)}{n}} \overset{n \gg \log(n+1)}{\longrightarrow} e^0 = 1$. Therefore

$$\sum_n |a_n|, \begin{cases} \text{converges,} & \text{if } |\log x| < 1, \iff -1 < \log x < 1, \iff e^{-1} < x < e, \\ \text{diverges and } |a_n| \to +\infty, & \text{if } |\log x| > 1, \iff \log x < -1, \vee \log x > 1, \iff x < e^{-1}, \vee x > e. \end{cases}$$

In the cases $|\log x| = 1$ (that is $x = e^{-1}, e$) test fails. For the moment, we can say that

$$\sum_n a_n, \begin{cases} \text{absolutely (hence simply) convergent,} & \text{if } \frac{1}{e} < x < e, \\ \text{non convergent (simply or absolutely) being } a_n \not\longrightarrow 0, & \text{if } x < \frac{1}{e}, \vee x > e. \end{cases}$$

To finish we have to study the cases $x = \frac{1}{e}, e$. Replacing directly these values,

- if $x = \frac{1}{e} = e^{-1}$ the series is

  $$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n(n+1)}},$$

  that is an alternate sign terms series. Being $\frac{1}{\sqrt{n(n+1)}} \searrow 0$ the series converges by Leibniz test. Does it converge also absolutely? We have

  $$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n(n+1)}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} \text{ and } \frac{1}{\sqrt{n(n+1)}} = \frac{1}{n\sqrt{1+\frac{1}{n}}} \sim \frac{1}{n}, \overset{asymp.\ comp.}{\Longrightarrow} \quad \text{diverges.}$$

- if $x = e$ the series becomes

  $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}},$$

  which is of course simply and absolutely (it is the same!) divergent from previous point.

Conclusion: the series is simply convergent as $x \in \left[\frac{1}{e}, e\right[$, absolutely convergent as $x \in \left]\frac{1}{e}, e\right[$. □

### 9.5. Decimal Representation of Reals

As everybody knows, naturals and integers are represented by *decimal digits*. By this we obtain a representation of rationals. What about reals? To fix ideas, let $x \in \mathbb{R}$ be a positive number and let $[x]$ be its integer part in such a way that

$$[x] \leqslant x < [x] + 1.$$

Dividing the interval $[[x], [x] + 1]$ into 10 equal parts we have that $x$ will belong to just one of these. Formally,

$$\exists! k_1 \in \{0, 1, 2, \dots, 9\} \ : \ [x] + \frac{k_1}{10} \leqslant x < [x] + \frac{k_1 + 1}{10}.$$

Repeating this argument to the interval $\left[[x] + \frac{k_1}{10}, [x] + \frac{k_1+1}{10}\right]$, that is dividing this into ten equal parts, our $x$ will belong to just one of these:

$$\exists! k_2 \in \{0, 1, 2, \dots, 9\} \ : \ [x] + \frac{k_1}{10} + \frac{k_2}{10^2} \leqslant x < [x] + \frac{k_1}{10} + \frac{k_2 + 1}{10^2}.$$

Iterating this procedure, we may find

$$\exists! k_1, k_2, \dots, k_n \in \{0, 1, 2, \dots, 9\} \ : \ [x] + \sum_{j=1}^{n-1} \frac{k_j}{10^j} + \frac{k_n}{10^n} \leqslant x < [x] + \sum_{j=1}^{n-1} \frac{k_j}{10^j} + \frac{k_n + 1}{10^n}.$$

In other words, we proved that *for every* $x \geqslant 0$ *there exists a* **unique** *infinite list of digits* $k_j \in \{0, 1, 2, \dots, 9\}$ *such that*

$$0 \leqslant x - \left([x] + \sum_{j=1}^{n} \frac{k_j}{10^j}\right) < \frac{1}{10^n}. \tag{9.5.1}$$

Letting $n \longrightarrow +\infty$, we draw

$$x = [x] + \sum_{j=1}^{\infty} \frac{k_j}{10^j}. \tag{9.5.2}$$

We identify (9.5.2) with the following writing:

$$x \equiv [x], k_1 k_2 k_3 \dots$$

We call this last *decimal decomposition of* $x$. We can easily extend this to negative numbers: if $x < 0$ then $-x > 0$ thus

$$-x \equiv [-x], k_1 k_2 k_3 \dots,$$

by which

$$x \equiv -[-x], k_1 k_2 k_3 \dots.$$

As you can see, what we call a *real number* is nothing but an infinite series. This should make clear how relevant are the series for Mathematics.

## 9.6. Exercises

Exercise 9.6.1 (★). For any of the following series, compute the partial sums and discuss the convergence (computing the eventual sum) of the series:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n-1)(2n+1)}$. 2. $\displaystyle\sum_{n=0}^{\infty} \left(\sqrt{n(n+1)} - \sqrt{n(n-1)} - 1\right)$. 3. $\displaystyle\sum_{n=2}^{\infty} \log\left(1 - \frac{1}{n^2}\right)$.

Exercise 9.6.2. For any of the following series $\sum_n a_n$ find $b_n$ such that $a_n \leqslant b_n$ and $\sum_n b_n$ converges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3+(-1)^n}}$. 2. $\displaystyle\sum_{n=0}^{\infty} \frac{1+\sin n}{2^n}$. 3. $\displaystyle\sum_{n=0}^{\infty} 2^{\sqrt{n}} 3^{-n}$.

Exercise 9.6.3. Applying the asymptotic comparison test, we discuss the convergence of the following series

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)^n}{n^{n+2}}$. 2. $\displaystyle\sum_{n=1}^{\infty} \frac{n+\log n}{(n-\log n)^3}$. 3. $\displaystyle\sum_{n=0}^{\infty} (\sqrt[3]{n+1} - \sqrt[3]{n})$.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$. 5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{n+2}{n+3}\right)^n$. 6. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-n}(\sqrt{n+1}-\sqrt{n})}$.

7. $\displaystyle\sum_{n=0}^{\infty} n^\beta \left(1 - \sqrt{\frac{n^3}{n^3+1}}\right)$, $(\beta \in \mathbb{R})$.

Exercise 9.6.4. Applying root or ratio tests, determine if the following series converge:

1. $\displaystyle\sum_{n=0}^{\infty} 5^{(-1)^n - n}$. 2. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{n^2(2n)!}$. 3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^k}{n!}$. 4. $\displaystyle\sum_{n=0}^{\infty} \frac{n^{n+1}}{3^n(n+1)!}$.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{nx}}{n!}$, $(x \in \mathbb{R})$. 6. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^x}{n^n}$, $(x \in \mathbb{R})$. 7. $\displaystyle\sum_{n=1}^{\infty} x^{n!}$, $(x \geqslant 0)$. 8. $\displaystyle\sum_{n=0}^{\infty} n!\, x^n$, $(x \geqslant 0)$.

Exercise 9.6.5. Applying Leibniz test, determine if the following series converge:

1. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$. 2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n+1}-\sqrt{n}}{n}$. 3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n (\sqrt[n]{3} - 1)$.

4. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(1 - \cos\frac{1}{\sqrt{n+1}}\right)$. 5. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}} - 1}{\cos(n\pi)}$. 6. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{1+\frac{1}{n}} - 1\right)$.

7. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\frac{1}{n}$. 8. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{n}\right)^n$. 9. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{4^n}$.

10.(★) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2}{n^2(2n)!}$. 11.(★) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$.

Exercise 9.6.6. For any of the following series, determine simple and absolute convergence.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}$, $(x\in\mathbb{R})$.
2. $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n^2+e^x}$, $(x\in\mathbb{R})$.
3. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x^2+1}{x^2-4}\right)^{2n+n^2}$, $(x\in\mathbb{R}\backslash\{\pm 2\})$.
4. $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{1+x^{2n}}$, $(x\in\mathbb{R})$.
5. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2+1}(x^2-1)^{n^2}$, $(x\in\mathbb{R})$.
6. $\displaystyle\sum_{n=1}^{\infty}\frac{(\sin x)^n}{n+\sqrt{n}\log n}$, $(x\in\mathbb{R})$.
7. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt[3]{1-x^{2n}}}{3^n}$ $(x\in\mathbb{R})$.
8. $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n+\arctan n}$, $(x\in\mathbb{R})$
9. $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)}\left(\frac{e^x+1}{e^x-1}\right)^n$, $x\neq 0$.

Exercise 9.6.7. Prove that if $\sum_n a_n$ and $\sum_n b_n$ converge then also $\sum_n(a_n+b_n)$ converges and

$$\sum_n(a_n+b_n)=\sum_n a_n+\sum_n b_n.$$

It it true that, if $\sum_n(a_n+b_n)$ converges then, necessarily, also $\sum_n a_n$ and $\sum_n b_n$ converge?

Exercise 9.6.8. Let $(a_n)\subset]0,+\infty[$. Show that

$$\sum_n a_n \text{ converges} \quad\Longleftrightarrow\quad \sum_n\frac{a_n}{1+a_n} \text{ converges.}$$

(hint: use asymptotic comparison, justifying carefully all steps. . . )

Exercise 9.6.9. Discuss the convergence for the series

$$\sum_{n=2}^{\infty}\frac{1}{n^\alpha(\log n)^\beta},$$

where $\alpha,\beta>0$.