Methods and Models for Combinatorial Optimization

The assignment problem and totally unimodular matrices

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1 The assignment problem

Let $G = (V, E)$ be a bipartite graph, where V is the vertex set and E is the edge set. Recall that "bipartite" means that V can be partitioned into two disjoint subsets V_1, V_2 such that, for every edge $uv \in E$, one of u and v is in V_1 and the other is in V_2.

In the assignment problem we have $|V_1| = |V_2|$ and there is a cost c_{uv} for every edge $uv \in E$. We want to select a subset of edges such that every node is the end of exactly one selected edge, and the sum of the costs of the selected edges is minimized. This problem is called "assignment problem" because selecting edges with the above property can be interpreted as assigning each node in V_1 to exactly one adjacent node in V_2 and vice versa.

To model this problem, we define a binary variable x_{uv} for every $u \in V_1$ and $v \in V_2$ such that $uv \in E$, where

$$x_{uv} = \begin{cases} 1 & \text{if } u \text{ is assigned to } v \text{ (i.e., edge } uv \text{ is selected),} \\ 0 & \text{otherwise.} \end{cases}$$

The total cost of the assignment is given by

$$\sum_{uv \in E} c_{uv} x_{uv}.$$

The condition that for every node u in V_1 exactly one adjacent node $v \in V_2$ is assigned to v_1 can be modeled with the constraint

$$\sum_{v \in V_2 : uv \in E} x_{uv} = 1, \quad u \in V_1,$$

while the condition that for every node v in V_2 exactly one adjacent node $u \in V_1$ is assigned to v_2 can be modeled with the constraint

$$\sum_{u \in V_1 : uv \in E} x_{uv} = 1, \quad v \in V_2.$$

The assignment problem can then be formulated as the following integer linear programming problem:

$$
\begin{array}{rcll}
\min \sum_{uv \in E} c_{uv} x_{uv} & & & \\
\sum_{v \in V_2 : uv \in E} x_{uv} & = & 1, & u \in V_1, \\
\sum_{u \in V_1 : uv \in E} x_{uv} & = & 1, & v \in V_2, \\
x_{uv} & \geq & 0, & uv \in E, \\
x_{uv} & \in & \mathbb{Z}, & uv \in E.
\end{array} \tag{1}
$$

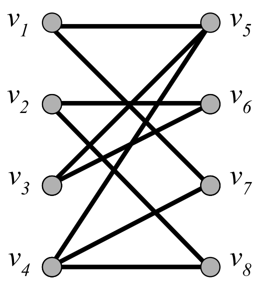
Note that we can omit the inequalities $x_{uv} \leq 1$ for every $uv \in E$, as they are implied by the other constraints.

In the following we will write the assignment problem in matrix form.

Definition 1 *Given an undirected graph $G = (V, E)$ (not necessarily bipartite), the* incidence matrix of G *is the matrix $A(G)$ with $|V|$ rows, $|E|$ columns and all entries in $\{0, 1\}$, where the element $a_{v,e}$ (i.e, the element in the row of $A(G)$ corresponding to node $v \in V$ and in the column of $A(G)$ corresponding to edge $e \in E$) is*

$$
a_{v,e} = \begin{cases} 1 & \textit{if } v \textit{ is an end-node of } e, \\ 0 & \textit{otherwise.} \end{cases}
$$

Note that every column of $A(G)$ has exactly two entries of value 1 and all other entries equal to 0. Moreover, if G is bipartite then every column of $A(G)$ has exactly one entry of value 1 in the rows corresponding to nodes in V_1, and one entry of value 1 in the rows corresponding to nodes in V_2. A bipartite graph and its incidence matrix are shown below.



	v_1v_5	v_1v_7	v_2v_6	v_2v_8	v_3v_5	v_3v_6	v_4v_5	v_4v_7	v_4v_8
v_1	1	1	0	0	0	0	0	0	0
v_2	0	0	1	1	0	0	0	0	0
v_3	0	0	0	0	1	1	0	0	0
v_4	0	0	0	0	0	0	1	1	1
v_5	1	0	0	0	1	0	1	0	0
v_6	0	0	1	0	0	1	0	0	0
v_7	0	1	0	0	0	0	0	1	0
v_8	0	0	0	1	0	0	0	0	1

It can be easily checked that the assignment problem can be written in the form

$$
\begin{gathered}
\min c^T x \\
A(G)\, x = \mathbf{1} \\
x \geq 0 \\
x \in \mathbb{Z}^{|E|},
\end{gathered}
$$

where **1** denotes a vector whose components are all equal to 1.

In the following, we will show that in the above problem the integrality constraints can be removed, as the formulation is ideal. In other words, we will show that all feasible basic solutions of the system $A(G)\, x = \mathbf{1}$, $x \geq 0$, have integer components. This will follow from a fundamental property of incidence matrices of bipartite graphs: total unimodularity.

2 Totally unimodular matrices

We call a vector (or a matrix) *integer* if all its entries are integer.

Definition 2 *A matrix A is* totally unimodular *if* $\det(B) \in \{0, +1, -1\}$ *for every square submatrix B of A.*

In particular, since every entry of A is a 1×1 square submatrix of A, the entries of every totally unimodular matrix are 0, +1 and −1.

We recall a basic fact from linear algebra. Given a square matrix $B \in \mathbb{R}^{m \times m}$ and given two indices $i, j \in \{1, \ldots, m\}$, let B^{ji} denote the matrix obtained from B by removing the j-th row and the i-th column. If B is invertible (i.e., $\det(B) \neq 0$), then the entry in position (i, j) of the inverse matrix is

$$(B^{-1})_{i,j} = (-1)^{i+j} \frac{\det(B^{ji})}{\det(B)}.$$

This implies that if B is integer then every entry of B^{-1} is an integer number divided by the determinant of B. Therefore, if B is integer and $\det(B) = \pm 1$, B^{-1} is also an integer matrix. In particular, if A is a totally unimodular matrix, then for every invertible square submatrix B of A the matrix B^{-1} is integer.

Theorem 1 *Let $A \in \mathbb{R}^{m \times n}$ be a totally unimodular matrix and $b \in \mathbb{R}^m$ be a integer vector. Then all basic solutions of the system*

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

are integer.

Proof: Given a basis B of A, the corresponding basic solution $\bar{x}$ is given by

$$\begin{aligned} \bar{x}_B &= B^{-1} b, \\ \bar{x}_N &= 0. \end{aligned}$$

Since A is totally unimodular, from the above discussion we know that B^{-1} is an integer matrix. Since b is an integer vector by assumption, $B^{-1}b$ is an integer vector and thus $\bar{x}$ is an integer vector. □

Theorem 2 *Let A be a matrix with all entries in $\{0, +1\}$ such that every column of A has at most two entries of value 1. If the rows of A can be partitioned into two sets V_1 and V_2 such that every column of A has at most one entry of value 1 in the rows of V_1 and at most one entry of value 1 in the rows of V_2, then A is totally unimodular.*

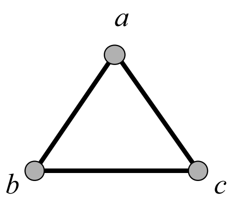
Proof: Let B be a $k \times k$ square submatrix of A. We will show by induction on k that $\det(B) \in \{0, +1, -1\}$. If $k = 1$, then B has a single entry that, by assumption, is 0 or 1, and therefore $\det(B) \in \{0, 1\}$. Now take $k \geq 2$ and assume by induction that every $(k-1) \times (k-1)$ square submatrix of A has determinant 0, +1 or −1. Note that every column of B has at most two entries of value 1. We consider three cases.
a) B has at least one all-zero column. In this case $\det(B) = 0$.
b) B has at least one column with exactly one entry equal to 1. Let us assume that the j-th column of B has a single entry of value 1, say the entry in row i. By Laplace rule, if B' is the submatrix of B obtained by removing the i-th row and the j-th column, then $\det(B) = (-1)^{i+j} \det(B')$. By induction, $\det(B') \in \{0, +1, -1\}$, and therefore $\det(B) = (-1)^{i+j} \det(B') \in \{0, +1, -1\}$.
c) Each column of B has precisely two entries of value 1. In this case, by assumption every column of A has one entry of value 1 in the rows of V_1 and one entry of value 1 in the rows of V_2. Then the sum of the rows of B in V_1 minus the sum of the rows of B in V_2 is the zero vector. This implies that the rows of B are linearly dependent, and therefore $\det(B) = 0$. □

Corollary 1 *The incidence matrix of a bipartite graph is totally unimodular.*

Proof: Let $G = (V, E)$ be a bipartite graph and let V_1, V_2 be the partition of the nodes of G such that every edge has one end in V_1 and the other in V_2. Then every column of $A(G)$ has exactly one entry equal to 1 in the rows corresponding to nodes in V_1 and exactly one entry equal to 1 in the rows corresponding to nodes in V_2 . Then $A(G)$ is totally unimodular by Theorem 2. □

By Theorem 1 and Corollary 1, all basic solutions of (1) are integer, and therefore the assignment problem can be solved with the simplex method.

We remark that in Corollary 1 the assumption that the graph is bipartite is essential. For instance, the incidence matrix of the graph below has determinant −2 and thus it is not totally unimodular.



	ab	ac	bc
a	1	1	0
b	1	0	1
c	0	1	1

Several operations preserve totally unimodularity. For instance, if A is an $m \times n$ totally unimodular matrix, then

(1) every matrix obtained from A by permuting some rows and/or columns is totally unimodular,

(2) every matrix obtained from A by multiplying some rows and/or columns by -1 is totally unimodular,

(3) A^T is totally unimodular,

(4) the matrix (A, I) is totally unimodular (where I is the $m \times m$ identity matrix),

(5) the matrix $\begin{pmatrix} A \\ I \end{pmatrix}$ is totally unimodular (where I is the $n \times n$ identity matrix).

Facts (1) and (2) are immediate, as if B is a square matrix and B' is obtained from B by permutation of rows and columns or by multiplying some rows or columns by -1, then $\det(B') = \pm \det(B)$. Property (3) holds because $\det(B) = \det(B^T)$. Fact (5) follows from (3) and (4). To show that (4) holds, let B be a square submatrix of (A, I). Up to row permutations, we can write B in the form

$$\begin{pmatrix} C & 0 \\ D & I \end{pmatrix}$$

where C and D are submatrix of A and 0 denotes an all-zero matrix. Then, by basic linear algebra, $\det(B) = \det(C) \in \{0, +1, -1\}$ because C is a submatrix of the totally unimodular matrix A.

The above facts and Theorem 1 imply the following result.

Theorem 3 *Let $A \in \mathbb{R}^{m \times n}$ be a totally unimodular matrix and $b \in \mathbb{R}^m$ be an integer vector. Then all basic solutions of the system*

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned}$$

are integer.

Proof: A vector $\bar{x}$ is a basic solution of the above system if the vector $(\bar{x}, \bar{s})$, where $\bar{s} = b - A\bar{x}$, is a basic solution of the equivalent system in standard form

$$\begin{aligned} Ax + Is &= b \\ x \geq 0,\ s &\geq 0. \end{aligned}$$

By property 4, the matrix of the system of equations (A, I) is totally unimodular. Then, by Theorem 1, $(\bar{x}, \bar{s})$ is an integer vector. In particular, $\bar{x}$ is an integer vector. $\square$

The transportation problem Let $G = (V, E)$ be an undirected bipartite graph, and let V_1, V_2 be a partition of V such that every edge of G has one end in V_1 and the other in V_2. For every node $u \in V_1$, let $d_u \in \mathbb{Z}$ be the amount that can be sent from u to the nodes in V_2, and for every node $v \in V_2$ let $r_v \in \mathbb{Z}$ be the amount required by node v. For every edge $uv \in E$ (with $u \in V_1$ and $v \in V_2$), let c_{uv} be the cost for transporting one unit from u to v. We want to plan a transportation from the nodes in V_1 to the nodes in V_2 at the minimum total cost, satisfying the demands of the nodes in V_2 and in such a way that the amount sent from every node $u \in V_1$ does not exceed d_u. This is called the *transportation problem* and can be formulated as

$$
\begin{array}{rcll}
\min \sum_{uv \in E} c_{uv} x_{uv} & & & \\
\sum_{v \in V_2 : uv \in E} x_{uv} & \leq & d_u, & u \in V_1, \\
\sum_{u \in V_1 : uv \in E} x_{uv} & \geq & r_v, & v \in V_2, \\
x_{uv} & \geq & 0, & uv \in E, \\
x_{uv} & \in & \mathbb{Z}, & uv \in E.
\end{array}
\tag{2}
$$

where x_{uv} is the amount sent from $u \in V_1$ to $v \in V_2$.

By Theorems 2 and 3, all basic solutions of the above system are integer and therefore the linear relaxation already provides an integer optimal solution.

Incidence matrices of directed graphs Given a directed graph $D = (V, A)$, the *incidence matrix* of D is the matrix $A(D)$ with entries in $\{0, +1, -1\}$ with $|V|$ rows and $|A|$ columns, where for every arc $e = (v, w)$ the element $a_{u,e}$ (in the row corresponding to node u and column corresponding to arc e) is

$$
a_{u,e} = \begin{cases} -1 & \text{if } u = v, \\ 1 & \text{if } u = w, \\ 0 & \text{otherwise.} \end{cases}
$$

Theorem 4 *The incidence matrix of every directed graph is totally unimodular.*

The proof of Theorem 4 is essentially the same as that of Theorem 2: the only difference is that, if B is a square submatrix of $A(D)$ in which every column of B has precisely two nonzero entries, then the sum of all the rows of B gives the all-zero vector (because every column contains exactly one $+1$ and one -1), therefore in this case the rows of B are linearly dependent, and thus $\det(B) = 0$.

The maximum flow problem As an application, consider the maximum flow problem. Given a directed graph $D = (V, A)$ with capacity c_{uv} for every arc $(u, v) \in A$, let $s, t \in V$ be two special nodes, called source and sink. There is no arc entering in the source s and no arc leaving the sink t. A feasible s-t flow is a vector $x \in \mathbb{R}^{|A|}$ such that:

(a) $0 \leq x_{uv} \leq c_{uv}$ for every $(u,v) \in A$ (i.e., the flow does not exceed arc capacities);

(b) for every node $u \in V \setminus \{s,t\}$,

$$\sum_{v \in V:(v,u) \in A} x_{vu} - \sum_{v \in V:(u,v) \in A} x_{uv} = 0$$

(i.e., in every node, except the source and the sink, the incoming flow equals the outgoing flow).

The value of the flow, which we denote by ϕ, is the total amount leaving node s, i.e.,

$$\phi = \sum_{v \in V:(s,v) \in A} x_{sv}.$$

(It can be shown that this value coincides with the total amount entering in node t.) The maximum flow problem is to find a feasible s-t flow of maximum value. The problem can be formulated as follows:

$$\begin{aligned}
& \max\ \phi && && (3)\\
\sum_{v \in V:(v,u) \in A} x_{vu} - \sum_{v \in V:(u,v) \in A} x_{uv} &= 0, && u \in V \setminus \{s,t\}, && (4)\\
-\sum_{v \in V:(s,v) \in A} x_{sv} + \phi &= 0, && && (5)\\
\sum_{v \in V:(v,t) \in A} x_{vt} - \phi &= 0, && && (6)\\
x_{uv} &\leq c_{uv}, && (u,v) \in A, && (7)\\
x_{uv} &\geq 0, && (u,v) \in A. && (8)
\end{aligned}$$

Constraints (4) ensure that condition (b) is satisfied; the other two equations define (twice, but this will be convenient) the objective function ϕ to be the value of the flow; constraints (7) and (8) enforce condition (a).

If we construct a directed graph D' by adding to D an arc from t to s (corresponding to variable ϕ), we can rewrite system (4), (5) and (6) in the form $A(D')\ x = 0$, where $A(D')$ is the incidence matrix of D' and therefore is totally unimodular by Theorem 4. Constraint (7) preserve totally unimodularity thanks to property (5). It follows that, when the capacities c_{uv} are all integer, by solving the above linear programming problem with the simplex method we find a maximum flow with integer components.[1]

[1]The form of the system is in part that of Theorem 1 and in part that of Theorem 3; however, similar to the proof of Theorem 3 one can show that all basic solutions are integer also for problems in this hybrid form.