Methods and Models for Combinatorial Optimization

Review of Duality in Linear Programming

L. De Giovanni M. Di Summa G. Zambelli

1 Definition of the dual problem

Duality theory in linear programming can be viewed as a tool for checking the optimality of a feasible solution. Given a linear programming problem in minimization form, the idea is to provide a lower bound on the possible values that the objective function can take over the feasible region.

Consider the following LP problem in standard form:

$$
\begin{array}{llrl}
(LP) & z^* = \min z = & c^T x & \\
 & \text{s.t.} & Ax & = b \\
 & & x & \geq 0
\end{array}
$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and x is a vector of variables in $\mathbb{R}^n$.

Definition 1 (Lower bound): *Given a LP problem $LP : \min\{c^T x : Ax = b, x \geq 0\}$, let z^* be the optimal value of the objective function. A number $\ell \in \mathbb{R}$ is called a lower bound for the problem if $\ell \leq c^T x$ for every x feasible for LP.*

One way to obtain a lower bound for (LP) is the following. Choose any vector $u \in \mathbb{R}^m$. If x is feasible for (LP), then

$$u^T Ax = u^T b.$$

If we also assume that $c^T \geq u^T A$, by using $x \geq 0$ we obtain:

$$c^T x \geq u^T Ax = u^T b, \ \forall\, x \text{ feasible}.$$

In other words: given a vector $u \in \mathbb{R}^m$, if $c \geq u^T A$ then $u^T b$ is a lower bound for (LP).

Note that, once a lower bound ℓ and a feasible solution $\tilde{x}$ are available, if $c^T \tilde{x} = \ell$ then $\tilde{x}$ is optimal. For this reason, it is convenient to look for the best possible lower bound, i.e., the largest lower bound. The best lower bound that one can obtain with this technique is given by a suitable choice of the vector u. We then have an optimization problem in which the decision variables are the entries of vector u:

$$
\begin{array}{llrl}
(DP) & \omega^* = \max \omega = & u^T b & \\
 & \text{s.t.} & u^T A & \leq c^T \\
 & & u & \text{free}
\end{array}
$$

Problem (DP) is the *dual problem* of (LP). In this context, (LP) is called *primal problem*, and the pair of problems (LP) and (DP) is called *primal-dual pair*.

Note that there is:

- a dual variable corresponding to each primal constraint;
- a dual constraint corresponding to each primal variable.

2 Duality theorems

The definition of dual problem implies the following result:

Theorem 1 (Weak duality) *If $\tilde{x}$ is a feasible solution for (LP) and $\tilde{u}$ is a feasible solution for (DP), then*

$$c^T \tilde{x} \geq \tilde{u}^T b.$$

It immediately follows that:

Corollary 1 *Given a feasible solution $\tilde{x}$ for (LP) and a feasible solution $\tilde{u}$ for (DP), if $c^T \tilde{x} = \tilde{u}^T b$ then $\tilde{x}$ is optimal for (LP) and $\tilde{u}$ is optimal for (DP).*

The following result can also be shown:

Corollary 2 *Given a primal-dual pair (LP)-(DP):*

(i) if (LP) is unbounded, then (DP) is infeasible;

(ii) if (DP) is unbounded, then (LP) is infeasible.

We remark that the above implications hold in the given direction only. Indeed, there are primal-dual pairs in which both problems are infeasible.

The following result also holds:

Theorem 2 (Strong duality): *(LP) has an optimal solution x^* if and only if (DP) has an optimal solution u^*. In this case, $c^T x^* = u^{*T} b$.*

We give a proof of one implication (namely, if (LP) has an optimal solution then (DP) has an optimal solution), as this is interesting to us because the theory of the simplex method is employed.

Proof: If (LP) has an optimal solution, then it has a basic optimal solution, which can be obtained by means of the simplex method. We denote this solution by $x^* = \begin{bmatrix} x_B^* \\ x_N^* \end{bmatrix}$, with $[B \mid N]$ being a decomposition of matrix A in its basic and non-basic columns. We now construct a suitable vector $u \in \mathbb{R}^m$ and show that it is feasible and optimal for (DP). Define

$$u^T = c_B^T B^{-1}.$$

Since x^* is an optimal solution given by the simplex method, we have

$$\bar{c}_B = 0$$
$$\bar{c}_N \geq 0$$

By the definition of reduced cost,

$$\bar{c}_N^T = c_N - c_B^T B^{-1} N = c_N - u^T N \geq 0 \Rightarrow c_N \geq u^T N.$$

We can then write

$$u^T A = u^T [B \mid N] = [u^T B \mid u^T N] = [c_B^T B^{-1} B \mid u^T N] = [c_B^T \mid u^T N] \leq [c_B^T \mid c_N^T] = c^T.$$

Then $u^T = c_B^T B^{-1}$ is a feasible solution for (DP). The corresponding value of the objective function is

$$u^T b = c_B^T B^{-1} b = c_B^T x_B^* = c^T x^*.$$

This shows that x^* and u form a pair of feasible solutions for (LP) and (DP) with the same value of the objective function. By Corollary 1, u is optimal for (DP). ■

The following table summarizes the above results:

		(DP)		
		Optimal	Unbounded	Infeasible
(LP)	Optimal	Possible (and $z^* = \omega^*$)	NO	NO
	Unbounded	NO	NO	Possible
	Infeasible	NO	Possible	Possible

3 Duality for problems in general form

The above arguments can be extended to LP problems not in standard form. The crucial point is the following: in order for $u^T b$ to be a lower bound for $c^T x$ ($u^T b \leq c^T x$), the following inequalities must be satisfied:

$$c^T x \geq u^T A x \geq u^T b.$$

For instance, if the constraints are of the form $Ax \geq b$, the second "$\geq$" is guaranteed to be true only if one imposes $u \geq 0$.
Similarly, the dual constraints on u that preserve the first "$\geq$" depend on the sign of x. By repeating the arguments for all possible cases (all possible directions of the inequalities and all possible signs of the variables), one obtains a table that summarizes the rules to construct the dual problem preserving the inequalities $c^T x \geq u^T A x \geq u^T b$.

Primal $(\min c^T x)$	**Dual** $(\max u^T b)$
$a_i^T x \geq b_i$	$u_i \geq 0$
$a_i^T x \leq b_i$	$u_i \leq 0$
$a_i^T x = b_i$	u_i free
$x_j \geq 0$	$u^T A_j \leq c_j$
$x_j \leq 0$	$u^T A_j \geq c_j$
x_j free	$u^T A_j = c_j$

The above table has to be read from left to right if the primal problem is a minimization one, and from right to left if the primal problem is a maximization one.

Note: all the results of the previous section can be generalized to any primal-dual pair, not necessarily in standard form.

Example:

$$\begin{array}{llllr} \min & 10x_1 & +20x_2 & & \\ s.t. & 2x_1 & -x_2 & & \geq 1 \\ & & x_2 & +x_3 & \leq 2 \\ & x_1 & & -2x_3 & = 3 \\ & & 3x_2 & -x_3 & \geq 4 \\ & x_1 & & & \geq 0 \\ & & x_2 & & \leq 0 \\ & & & x_3 & \text{free} \end{array} \Longrightarrow \begin{array}{llllll} \max & u_1 & +2u_2 & +3u_3 & +4u_4 & \\ s.t. & u_1 & & & & \geq 0 \\ & & u_2 & & & \leq 0 \\ & & & u_3 & & \text{free} \\ & & & & u_4 & \geq 0 \\ & 2u_1 & & +u_3 & & \leq 10 \\ & -u_1 & +u_2 & & +3u_4 & \geq 20 \\ & & u_2 & -2u_3 & -u_4 & = 0 \end{array}$$

4 Complementarity conditions

Consider a primal-dual pair:

$$\begin{array}{lll} (PP) & \min & c^T x \\ & \text{s.t.} & Ax \geq b \\ & & x \geq 0 \end{array} \qquad \begin{array}{lll} (DP) & \max & u^T b \\ & \text{s.t.} & u^T A \leq c^T \\ & & u \geq 0 \end{array} \qquad \begin{array}{ll} \text{and} & \text{vectors} \\ & \bar{x} \in \mathbb{R}^n \\ & \bar{u} \in \mathbb{R}^m \end{array}$$

The strong duality theorem implies the following *optimality conditions*:

$$\begin{array}{c} \bar{x} \text{ and } \bar{u} \text{ optimal for} \\ \text{primal and dual (resp.)} \end{array} \iff \begin{array}{ll} \bar{x} \text{ is primal feasible:} & A\bar{x} \geq b,\ \bar{x} \geq 0 \\ \bar{u} \text{ is dual feasible:} & \bar{u}^T A \leq c^T,\ \bar{u} \geq 0 \\ \text{strong duality holds:} & c^T \bar{x} = \bar{u}^T b \end{array}$$

Although we have used a primal problem with constraint of the form "$\geq$", the above conditions hold for problems in any form.

Exercise 1 *Find an optimal solution for the following problem:*

$$\begin{array}{lll} \max & z = & -3x_1 - x_3 \\ s.t. & & x_1 + 2x_2 + x_3 = 7 \\ & & 2x_1 + x_2 + x_3 = 20 \\ & & x_1,\ x_2,\ x_3 \textit{ free} \end{array}$$

Hint: the presence of just equality constraints and free variables suggests the direct application of strong duality to the problem and its dual; this yields a system of linear equations consisting of the primal constraints, the dual constraints (which are equality constraints because the primal variables are free) and the equality condition between the two objective functions.

Answer: infinitely many optimal solutions of the form $x_1 = 11 - \frac{1}{3}x_3$, $x_2 = -2 - \frac{1}{3}x_3$.

The optimality conditions can also be written as follows:

Theorem 3 (Complementary slackness (or orthogonality) conditions) *Given a primal-dual pair*

$$\begin{array}{l} \min\{c^T x : x \geq 0, Ax \geq b\} \\ \max\{u^T b : u \geq 0, u^T A \leq c^T\} \end{array}$$

$$\begin{array}{c} x \textit{ and } u \textit{ optimal for} \\ \textit{primal and dual (resp.)} \end{array} \iff \begin{array}{ll} Ax \geq b,\ x \geq 0 & \textit{(primal feasibility)} \\ u^T A \leq c^T,\ u \geq 0 & \textit{(dual feasibility)} \\ \left.\begin{array}{c} u^T(Ax - b) = 0 \\ (c^T - u^T A)x = 0 \end{array}\right\} & \textit{(complementarity)} \end{array}$$

More explicitly, the above complementarity conditions are the following:

$$\begin{array}{rcl} u^T(Ax - b) & = & \sum_{i=1}^{m} u_i(a_i^T x - b_i) = 0 \\ (c^T - u^T A)x & = & \sum_{j=1}^{n} (c_j - u^T A_j)x_j = 0 \end{array}$$

Since all the factors in the above expressions are ≥ 0 because of the feasibility of x and u,[1] we have:

[1] In this form, both factors of each summand are ≥ 0 and therefore each summand is ≥ 0. In general form, the domains of the dual variables and the directions of the dual constraints ensure that the two factors of each summands give a product ≥ 0, and therefore the above arguments hold in general.

$$\begin{aligned} u_i(a_i^T x - b_i) &= 0, \quad \boxed{\forall}\, i = 1, \ldots, m \\ (c_j - u^T A_j)x_j &= 0, \quad \boxed{\forall}\, j = 1, \ldots, n \end{aligned}$$

The complementarity (or orthogonality) conditions must be satisfied by all primal/dual variable/constraint pairs.

In other words, given feasible solutions x and u for a primal-dual pair $\min\{c^T x : x \geq 0, Ax \geq b\}$ and $\max\{u^T b : x \geq 0, u^T A \leq c^T\}$, x and u are optimal if and only if:

(1)	positive primal variable	$x_j > 0$	$\Rightarrow$	$u^T A_j = c_j$	tight dual constraint
(2)	strict dual constraint	$u^T A_j < c_j$	$\Rightarrow$	$x_j = 0$	primal variable $= 0$
(3)	positive dual variable	$u_i > 0$	$\Rightarrow$	$a_i^T x = b_i$	tight primal constraint
(4)	strict primal constraint	$a_i^T x > b_i$	$\Rightarrow$	$u_i = 0$	dual variable $= 0$

Note: the above implications hold in the given direction "$\Rightarrow$" only!

The above implications can be extended to primal-dual pairs in any form. For instance, if the primal is in standard form, conditions (3) and (4) are useless, since they directly comes from primal and/or dual feasibility.

5 The simplex method and duality

Consider an LP in standard form and the (partial) proof of the strong duality theorem given above. As we saw, given a basis B and its simplex multipliers $u^T = c_B^T B^{-1}$, the condition that "the reduced cost (with respect to B) of a variable is non-negative" is the same as the condition that "the corresponding dual constraint is satisfied by the dual solution given by the multipliers". In other words, given a basis, a non-negative (resp. negative) reduced cost of a primal variable corresponds to a dual constraint that is satisfied (resp. violated) by the multipliers associated to the basis itself. Indeed, for a given basis, the definition of non-negative reduced cost corresponds to that of the dual constraint associated to the multipliers:

$$\bar{c}_j = c_j - c_B^T B^{-1} A_j \geq 0 \Leftrightarrow c_j - u^T A_j \leq 0 \Leftrightarrow u^T A_j \leq c_j$$

Furthermore, the multipliers (viewed as a dual solution) and the current primal solution always satisfy the complementary slackness conditions by construction, as we now verify. Since the primal problem has only equality constraints (and non-negativity of the variables), the condition $u^T(Ax - b) = 0$ comes from primal feasibility. Since $x_B = B^{-1} b$ and $x_N = 0$, we have:

$$(c^T - u^T A)x = \big([c_B^T \mid c_N^T] - u^T[B \mid N]\big)[x_B \mid x_N] = c_B^T x_B + c_N^T x_N - c_B^T B^{-1} B x_B - c_B^T B^{-1} N x_N =$$
$$= c_B^T x_B + 0 - c_B^T x_B - 0 = 0.$$

In other words, $x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$ and $u = c_B^T B^{-1}$ are a pair of primal-dual solutions satisfying the complementary slackness conditions, and this is true *at every iteration of the simplex method*. The simplex method can thus be interpreted in two ways:

- as a method that, at every iteration, determines a feasible primal solution and tries to make it optimal;
- as a method that, at every iteration, determines a dual solution (the multipliers) satisfying the complementarity conditions with a feasible primal solution, and tries to make it feasible for the dual.

At termination, we will have a feasible primal solution and a feasible dual solution satisfying the complementarity conditions (and therefore optimal for the primal and the dual problem, respectively). During the execution of the simplex method, we instead have a primal-dual pair of solutions satisfying the complementarity conditions, but only the primal solution is feasible, therefore we can conclude the optimality of the solutions only when the algorithm terminates.

6 Exercises

Exercise 2 *Verify whether the solution $x_1 = 8$, $x_2 = 3$ is optimal for the following problem:*

$$\begin{array}{lrl} max & z = & 4x_1 + 2x_2 \\ s.t. & & 2x_1 \leq 16 \\ & & 2x_1 + 3x_2 \leq 25 \\ & & x_2 \leq 5 \\ & & x_1, x_2 \geq 0 \end{array}$$

Hint. After verifying that the given solution is feasible, write the dual and the complementarity slackness conditions. Use these conditions to find values for the dual variables, once the primal variables have been fixed at the given values. If (and only if) the values found for the dual variables are feasible for the dual problem, then the given primal solution is optimal, as in this case we have a primal-dual pair of feasible solutions satisfying the complementarity conditions, and therefore both optimal.

Answer: the given solution is optimal.

Exercise 3 *A company needs 11 kilograms of chromium, 2 of molybdenum and 1 of manganese. Two kinds of scrap steel can be purchased: one ton of the first type contains 3 kilograms of chromium and 1 of manganese, and costs 2000 euros; one ton of the second type contains 1 kilogram of chromium and 1 of molybdenum, and costs 3000 euros. Currently, the company purchases 3 and respectively 2 tons of first and the second type of scrap steel.*

1. *Verify that the current strategy is the cheapest one.*
2. *Check whether the optimal strategy may change after the availability of two new types of scrap steel. One ton of the third type contains 2 kilograms of chromium and 1 of manganese, and costs 1500 euros; one ton of the fourth type one contains 2 kilograms of chromium and 1 of molybdenum, and costs 4000 euro.*
3. *Check whether the optimal strategy may change after the availability of a new type of scrap steel containing 2 kilograms of chromium, 2 of molybdenum and 2 of manganese, and costing 5500 euro per ton.*

Resolution (sketch). We first write the model for the problem. We use variable x_i for the tons of scrap steel of type $i = 1, 2$ that should be purchased.

$$\begin{array}{lll}
\min \quad z = & 2x_1 + 3x_2 & \\
s.t. & 3x_1 + x_2 \geq 11 & (u_1) \\
& x_2 \geq 2 & (u_2) \\
& x_1 \geq 1 & (u_3) \\
& x_1, x_2 \geq 0 &
\end{array}$$

Solving point 1 simply means verifying that the solution $x_1 = 3$, $x_2 = 2$ is optimal (for instance, by using the complementarity conditions).

Answer to point 1: yes, the strategy is optimal.

In order to solve point 2, we note that the two new scrap types correspond to two new primal variables. Thus, in the dual problem we have two new constraints ($2u_1 + u_3 \leq 1.5$ and $2u_1 + u_2 \leq 4$). Since the dual now has more constraints, the optimal solution of the dual problem cannot improve: either it remains unchanged (if it does not violate the new constraints) or it deteriorates (if the previous optimal solution violates at least one new constraint). Therefore two cases are possible:

(a) the constraints are satisfied by the optimal dual variables found at point 1. Then the dual optimal solution is unchanged and, by strong duality, so is the primal optimal solution. This means that the availability of new scrap types does not affect the optimality of the current strategy.[2]

(b) the constraints are not satisfied by the optimal dual variables found at point 1. Then the optimal dual solution changes; in particular, it cannot increase, as there are new constraints (recall that the dual problem is a maximization one). Again by strong duality, the new primal optimal value will be equal to the new dual optimal value, and therefore smaller than (or equal to) the old one. It follows that the current strategy may improve.[3]

Answer to point 2: case (a).

The resolution of point 3 is similar; in this case the optimal strategy of the company changes.

Answer to point 3: case (b).

Exercise 4 *Given the problem*

$$\begin{array}{lll} max & z = & x_1 - x_2 \\ s.t. & & x_2 \leq 1 \\ & & 2x_1 + x_2 \leq 5 \\ & & -x_1 - 3x_2 \leq 10 \\ & & -x_1 - x_2 \leq 2 \\ & & x_1 \geq 0, x_2 \textit{ free} \end{array}$$

verify whether the solutions $x^a = [2, -4]$ *and* $x^b = [5, -5]$ *are optimal.*

[2] As an exercise, we give a more formal proof of the fact that if the new dual constraints are satisfied by the previous optimal dual solution u then the previous primal optimal solution is still optimal. We note that:

- the primal solution $x_1 = 3$, $x_2 = 2$, $x_3 = x_4 = 0$ is feasible also for the new problem, as the contribution of the new variables is 0;
- the number of dual variables is unchanged (because the number of primal constraints is still 3) and the dual solution u found at point 1 satisfies the complementarity conditions also with the new primal solution $x_1 = 3$, $x_2 = 2$, $x_3 = x_4 = 0$. This is because $x_3 \cdot$ (corresponding dual constraint) $= 0$ and $x_4 \cdot$(corresponding dual constraint) $= 0$; furthermore, $u_i \cdot$(corresponding new primal constraint) $= 0$ $\Leftrightarrow$ $u_i \cdot$ (corresponding old primal constraint) $= 0$ for $i = 1, 2, 3$, as the contribution of the new variables x_3 and x_4 is 0.

Therefore, if the dual constraints are satisfied, we have a primal-dual pair of feasible solutions satisfying the complementary slackness conditions, and therefore optimal.

[3] As an alternative proof, by using the more formal approach in footnote 2, we would have a pair of primal-dual solutions still satisfying the complementarity conditions, but the dual solution would not be feasible and therefore we cannot conclude that the two solutions are optimal.

We can again apply the complementarity conditions. Note that the second dual constraint will be an equation, as x_2 is a free variable. Thus it cannot be used to impose a complementarity conditions, but it can be used directly as one of the equations in the system that one writes to find a dual solution satisfying the complementarity conditions.

Answer: x^a is not optimal (the unique dual solution obtained by the system of equations violates the first dual constraints, so that there not exits any dual solution which is simultaneously feasible and satisfying complementarity conditions with respect to solution x^a); x^b is optimal (the unique dual solution obtained by the system of equalities is also dual-feasible).