Network Science

#3 Scale-free networks

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Why a network model?

Want to devise a model for generating a network that looks like a real one

- The model should capture meaningful network parameters
- The model should explain how networks emerge
- The model can be used to test our algorithms: many vs. few instances

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Random networks

Random networks

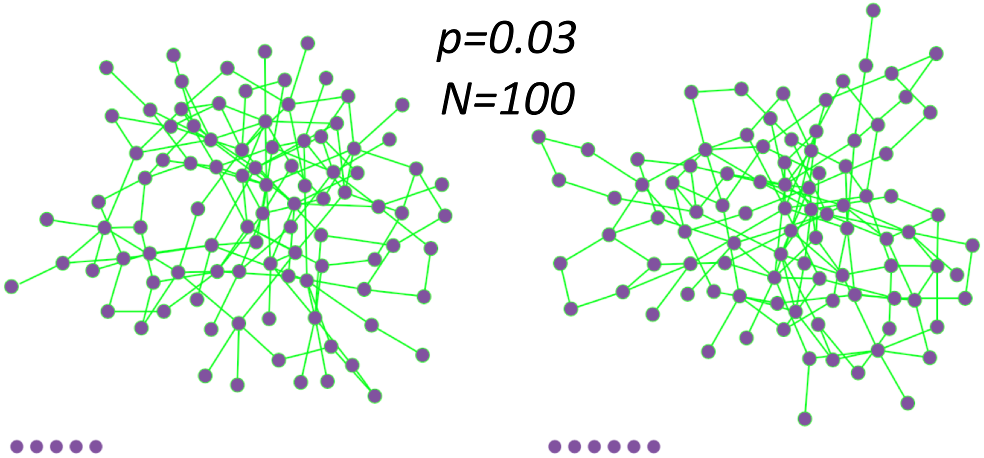
It seems that many connections arising in real networks are unpredictable

Idea:

- links are randomly generated, i.e., each link is active with probability p
- random = i.i.d. distributed

This seems sensible as we often observe unexpected links

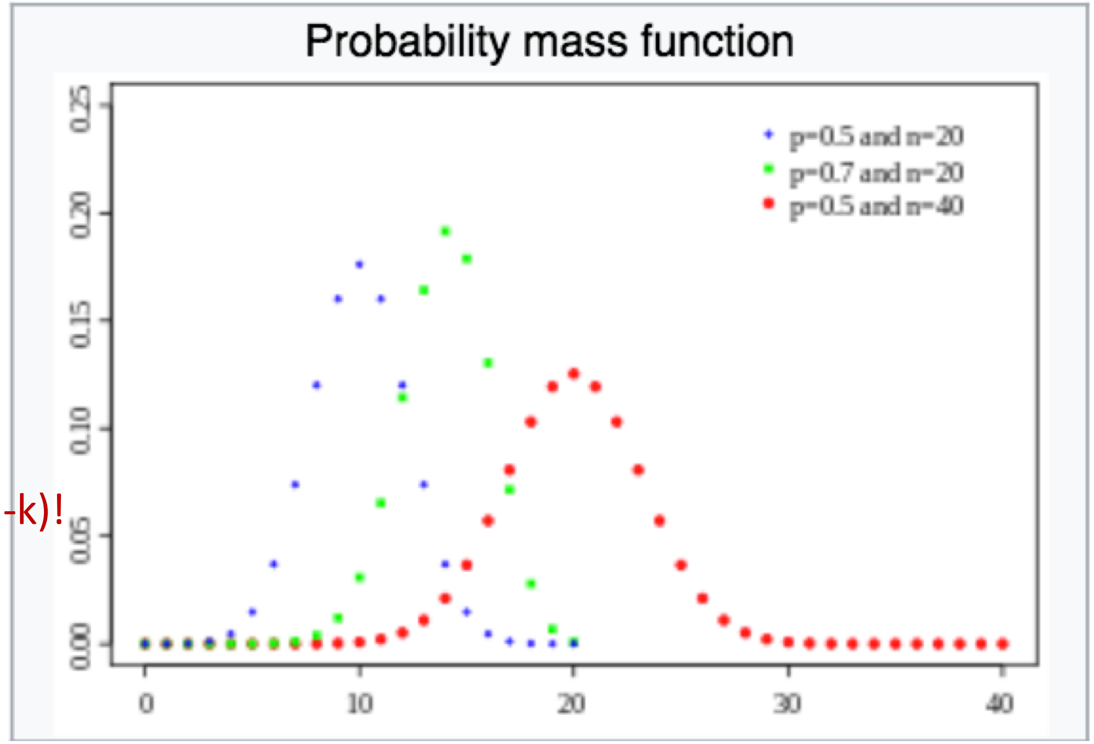
Erdös-Rényi model [1959/60]



- The random network is the simplest model:
  - pick a probability *p*, with *0<p<1*
  - activate each link (*i,j*) with probability p
- The number of links is variable
- There might be isolates
- Easy to calculate fundamental parameters

Binomial distribution

Notation	$B(n,p)$
Parameters	$n \in \{0,1,2,\ldots\}$ – number of trials $p \in [0,1]$ – success probability for each trial $q = 1 - p$
Support	$k \in \{0,1,\ldots,n\}$ – number of successes
PMF	$\binom{n}{k} p^k q^{n-k}$ ← binomial coefficient n!/k!(n-k)!
CDF	$I_q(n-k, 1+k)$
Mean	np
Median	$\lfloor np \rfloor$ or $\lceil np \rceil$
Mode	$\lfloor (n+1)p \rfloor$ or $\lceil (n+1)p \rceil - 1$
Variance	npq
Skewness	$\frac{q-p}{\sqrt{npq}}$
Ex. kurtosis	$\frac{1-6pq}{npq}$



P(k;n,p) = probability that *k* out of *n* trials are positive, where each is positive with probability *p*

Degree distribution

- The number of neighbours is binomially distributed

  $P(k;n,p)$ = probability that a node has exactly k neighbours, with number of possible neighbours $n = N-1$

- Average # of neighbours

  this defines p

  $$\langle k \rangle = (N-1)p \quad \rightarrow \quad p = \langle k \rangle / (N-1)$$

  p is usually very small (since $\langle k \rangle \ll N$)

- Variance

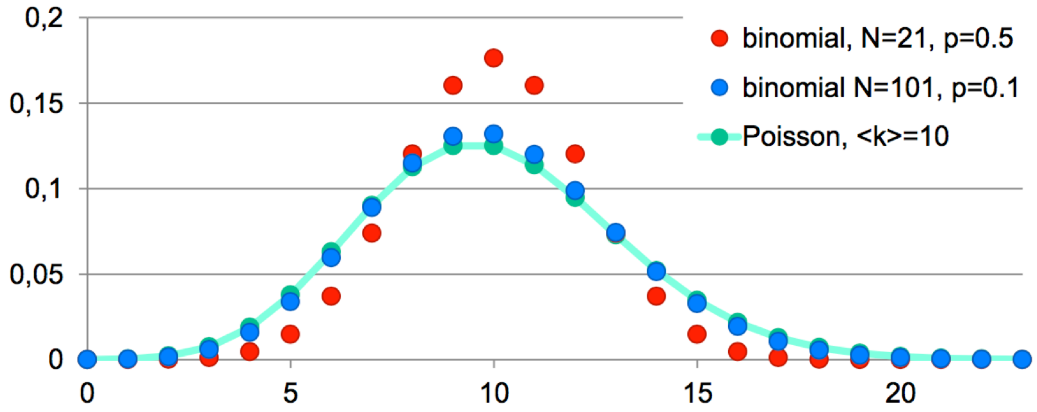
  $$\sigma_x^2 = (N-1)p(1-p) \simeq \langle k \rangle$$

  tight around the mean

Poisson approximation

- Poisson distribution (easier to use)

$$P[x = k] = \frac{m_x^k}{k!} \cdot e^{-m_x}$$

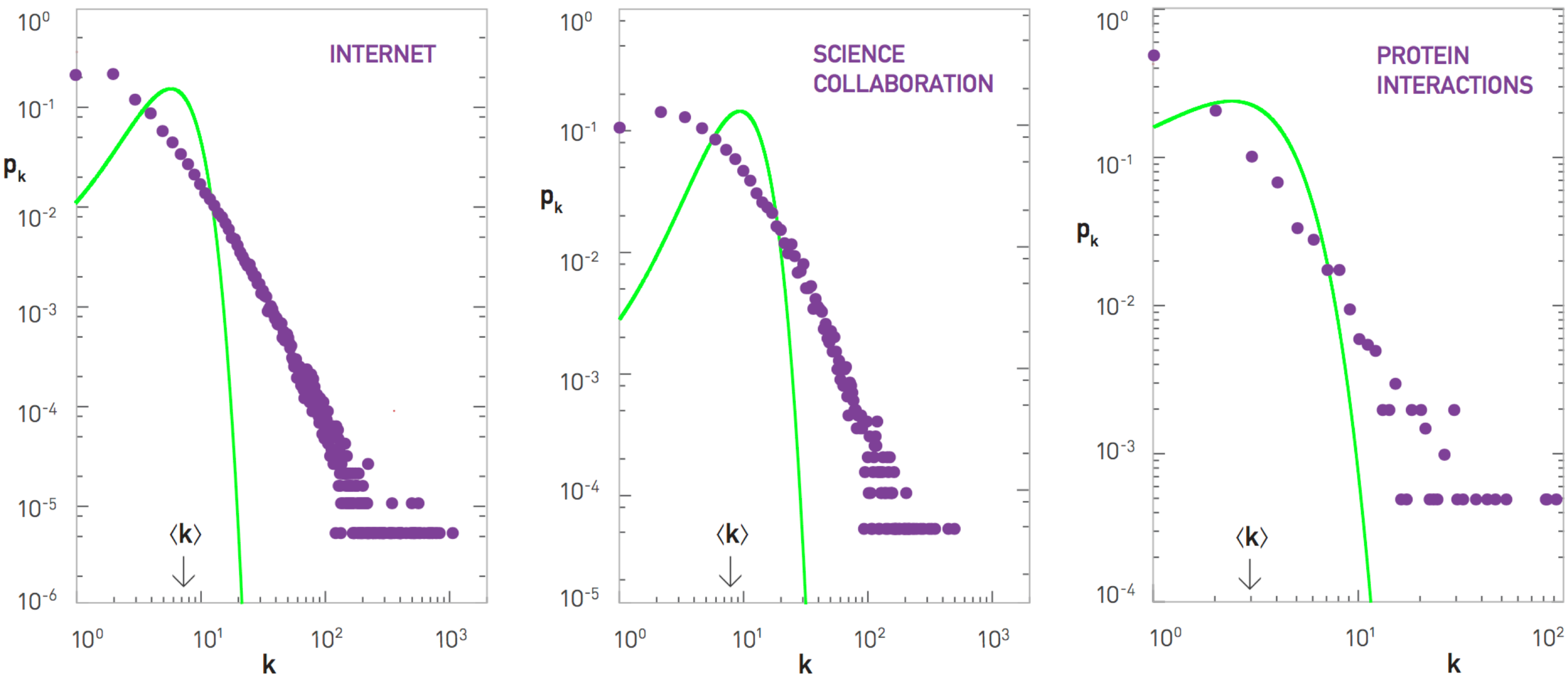


- Very good approximation of binomial for small *p* (and at small *k*)

active part   *p*

$$P[x = k] = \underbrace{\frac{(n-k+1)\dots(n-1)n}{n^k}}_{\simeq 1} \cdot \frac{m_x^k}{k!} \cdot \underbrace{\left(1 - \frac{m_x}{n}\right)^{n-k}}_{\simeq \text{const}}$$

Are real networks Poisson?

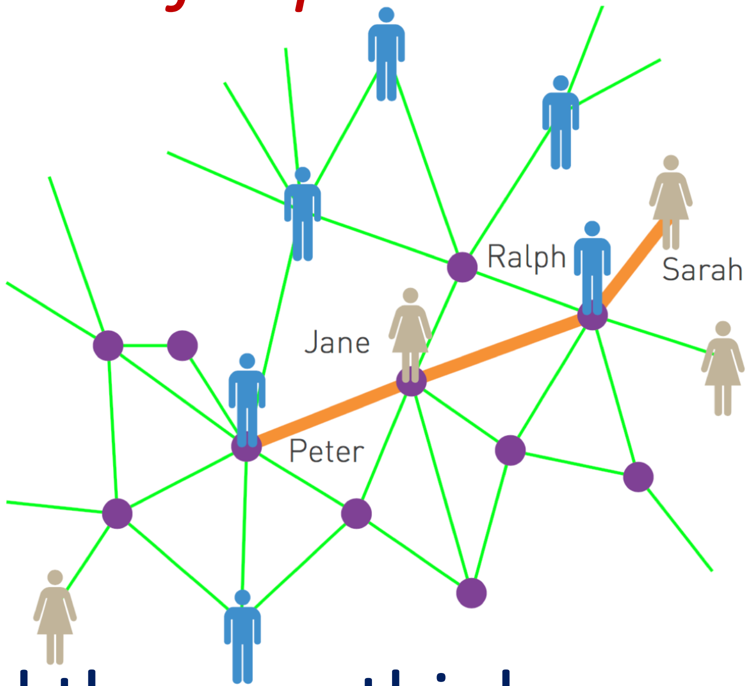


No! Poisson networks are deprived of hubs

… but, nevertheless, Poisson networks capture some aspects

Small world

- In real networks distance between two randomly chosen nodes is generally short
- Milgram [1967]: *6 degrees of separation*



- What does this mean?
  We are more connected than we think

Distances in random graphs

- we reach $\langle k \rangle$ nodes in one hop, $\langle k \rangle^2$ in two, $\langle k \rangle^3$ in three, etc.
- an estimate of the average distance $\langle d \rangle$ is found by solving for $N = \langle k \rangle^{\langle d \rangle}$ to have

$$\langle d \rangle = \ln(N)/\ln(\langle k \rangle)$$

- $\langle d \rangle$ is often taken as an estimate of the network diameter d_{max}

e.g.: on earth we are $N=7 \cdot 10^9$ individuals,
with $\langle k \rangle$=1000 acquaintances each → $\langle d \rangle = 3.28$

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Fit in real networks

NETWORK	N	L	$\langle k \rangle$	$\langle d \rangle$	d_{max}	$\frac{\ln N}{\ln \langle k \rangle}$
Internet	192,244	609,066	6.34	6.98	26	6.58 ✓
WWW	325,729	1,497,134	4.60	11.27	93	8.31 ✓
Power Grid	4,941	6,594	2.67	18.99	46	8.66
Mobile Phone Calls	36,595	91,826	2.51	11.72	39	11.42 ✓
Email	57,194	103,731	1.81	5.88	18	18.4
Science Collaboration	23,133	93,439	8.08	5.35	15	4.81 ✓
Actor Network	702,388	29,397,908	83,71	3,91	14	3,04 ✓
Citation Network	449,673	4,707,958	10.43	11,21	42	5.55
E. Coli Metabolism	1,039	5,802	5.58	2.98	8	4.04
Protein Interactions	2,018	2,930	2.90	5.61	14	7.14 ✓

Very good fit ! Correct at least as order of magnitude

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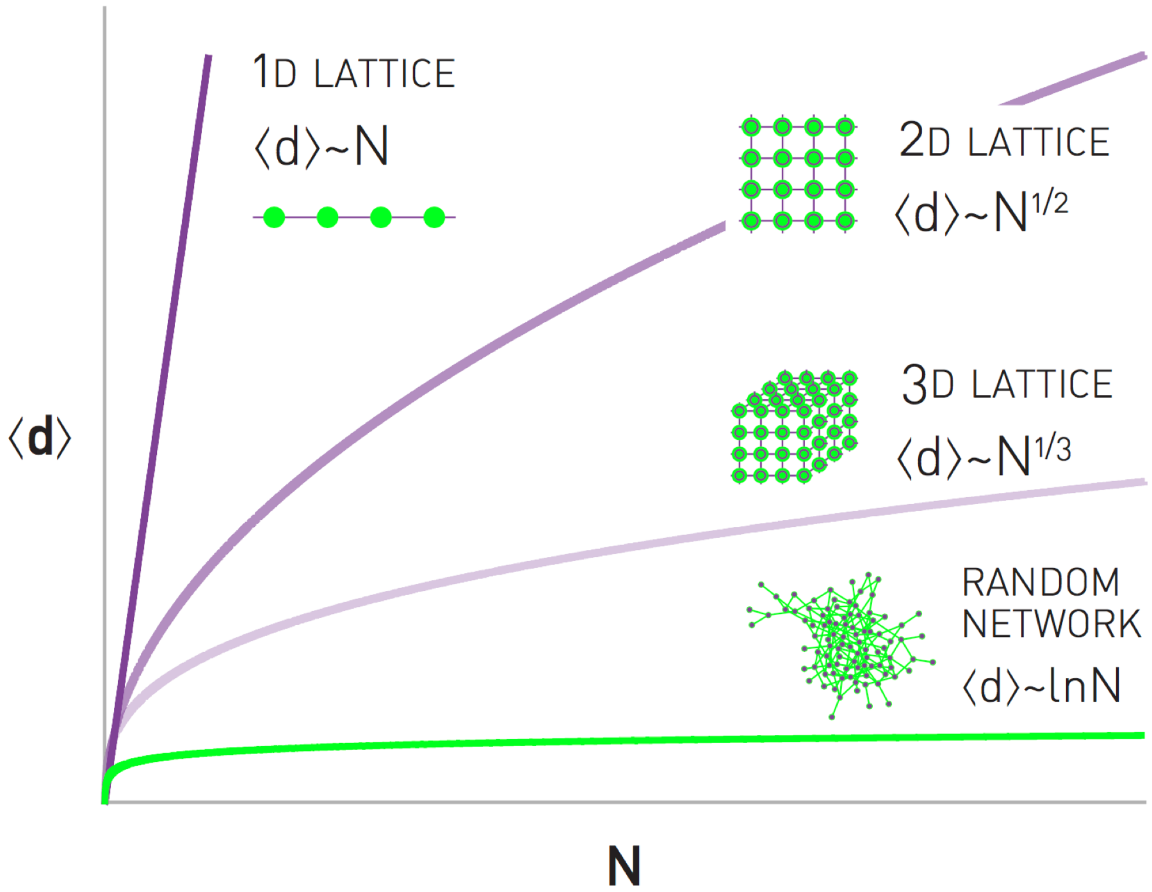
What about structured networks?



No fit !

Are real networks random?

- Random networks are generally NOT a good model for real world scenarios, but
  - ✓ Are easy to describe and generate
  - ✓ Can make some correct predictions
  - ✓ Can serve as a general reference model (in the sense that a model is good if it deviates from the random model as much as real networks do)

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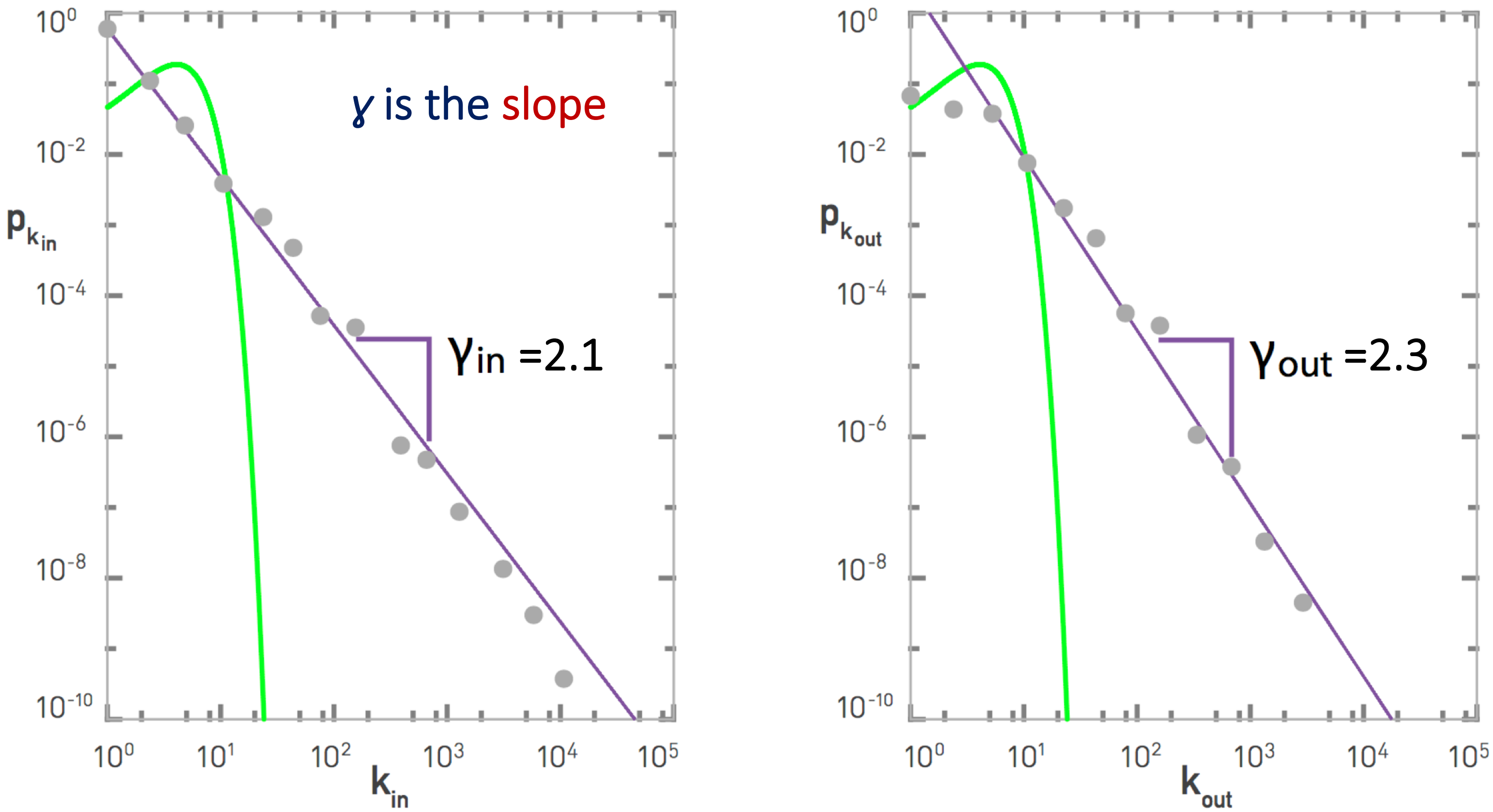
Scale-free networks

Scale free networks

The degree distribution of the www (Albert *et al.,* 1999)



they follow a power-law

$\ln(p_k) = c - \gamma \cdot \ln(k)$ → $p_k = C \cdot k^{-\gamma}$

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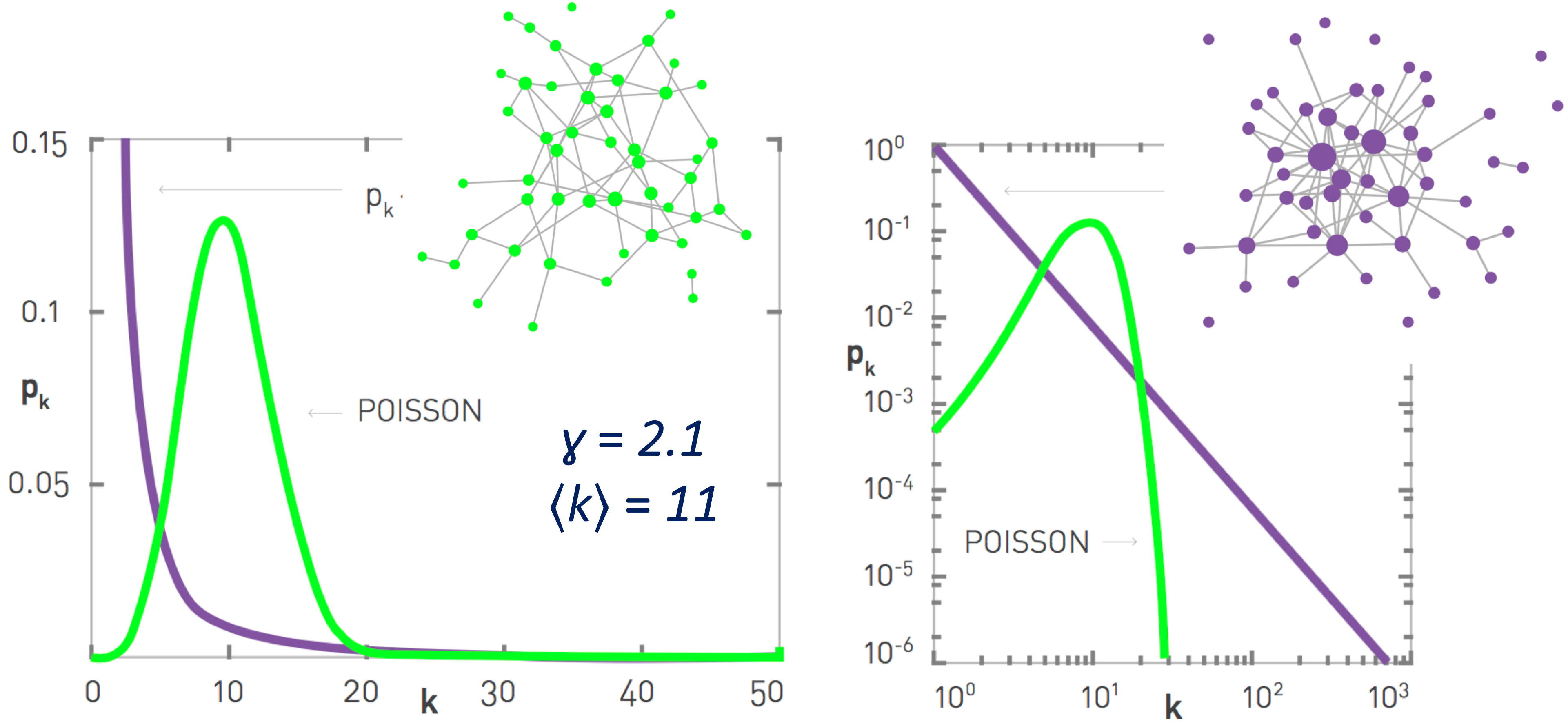
Poisson versus power-law



Power-law is heavy tailed (presence of hubs) - like Weibull, lognormal, Lévy

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The value of γ in real networks

NETWORK	N	L	$\langle k \rangle$	γ_{in}	γ_{out}	γ
Internet	192,244	609,066	6.34	-	-	3.42*
WWW	325,729	1,497,134	4.60	2.00	2.31	-
Power Grid	4,941	6,594	2.67	-	-	Exp.
Mobile Phone Calls	36,595	91,826	2.51	4.69*	5.01*	-
Email	57,194	103,731	1.81	3.43*	2.03*	-
Science Collaboration	23,133	93,439	8.08	-	-	3.35*
Actor Network	702,388	29,397,908	83.71	-	-	2.12*
Citation Network	449,673	4,689,479	10.43	3.03**	4.00*	-
E. Coli Metabolism	1,039	5,802	5.58	2.43*	2.9 0*	-
Protein Interactions	2,018	2,930	2.9 0	-	-	2.89*

$\gamma \in [2,4]$, but the statistical fit is not always good (** = good fit with an exponential)

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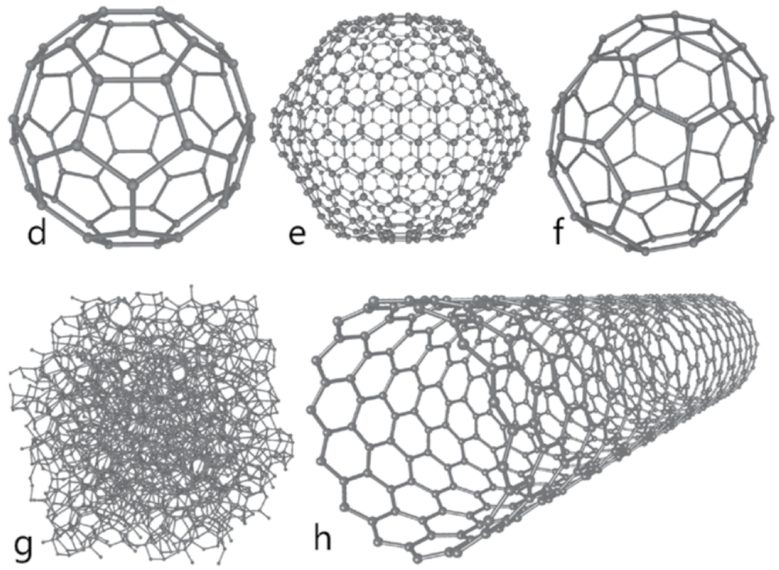
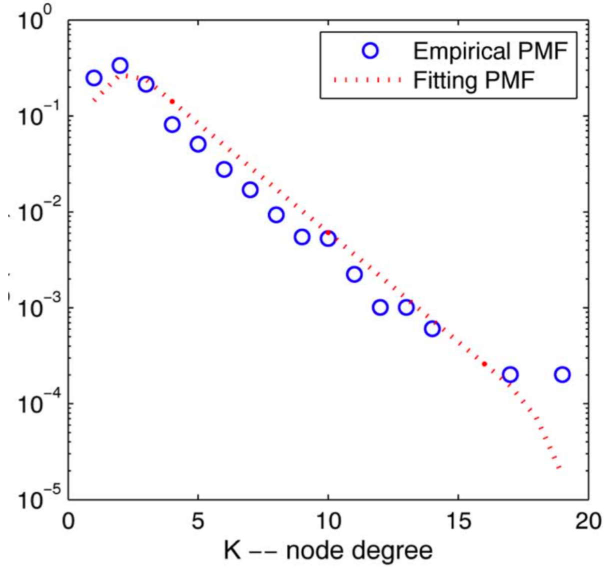
Is everything scale-free?

Not all networks are scale-free, e.g.:

- Networks appearing in material science, where all nodes have the same degree
- The neural network of the *C.elegans* worm
- The power grid, consisting of generators and switches connected by power lines





Scale-free networks

- A scale free network is a network whose degree distribution follows a power law $p_k = C \cdot k^{-\gamma}$

- It is meaningful in an interval $[k_{min}, k_{max}]$

- The parameter γ (slope) is called the exponent

- C is determined by the total normalization condition $\sum_{k=k_{min}}^{k_{max}} p_k = 1$ which we approximate to

$$\int_{k_{min}}^{\infty} p_k \, dk = C \cdot k_{min}^{-(\gamma-1)} / (\gamma - 1) = 1$$

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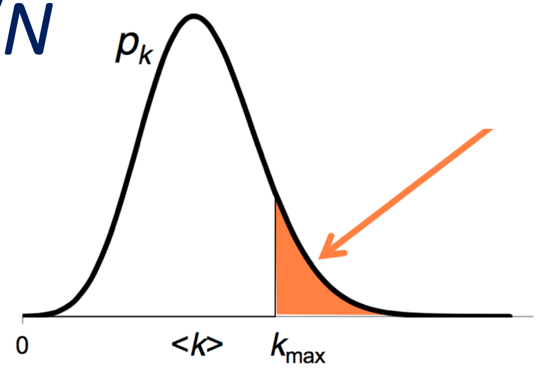
The largest hub

The size of the largest hub is captured by

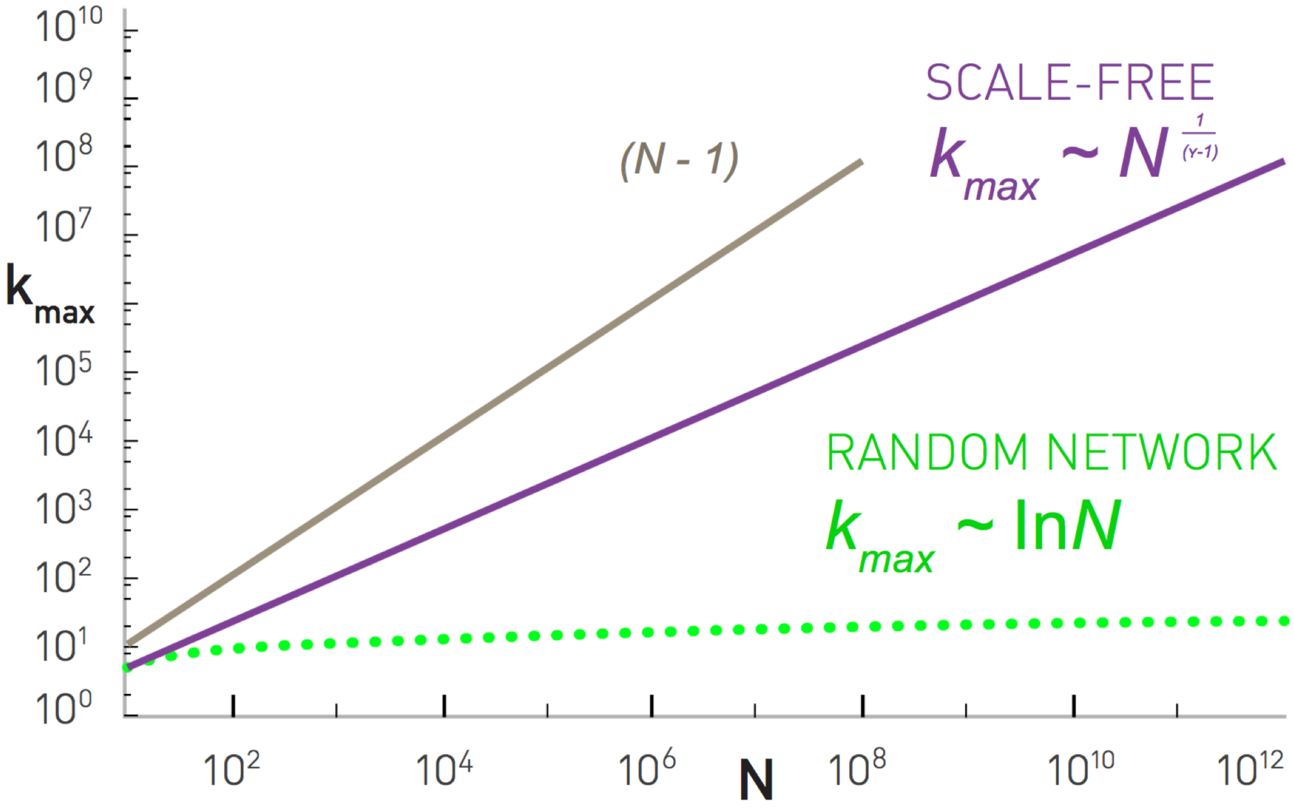
$$\int_{k_{max}}^{\infty} p_k \, dk = C \cdot k_{max}^{-(\gamma-1)} / (\gamma - 1) = 1/N$$



$$k_{max} = k_{min} N^{1/(\gamma-1)}$$

is the natural cutoff

Hubs are large in scale-free networks



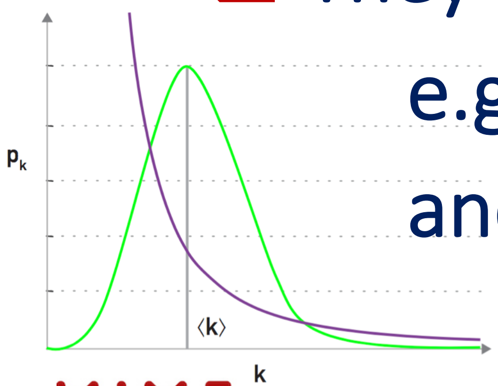
The highest degree increases in *N* polynomially (sub-linearly) fast = big hubs

Moments of scale-free networks

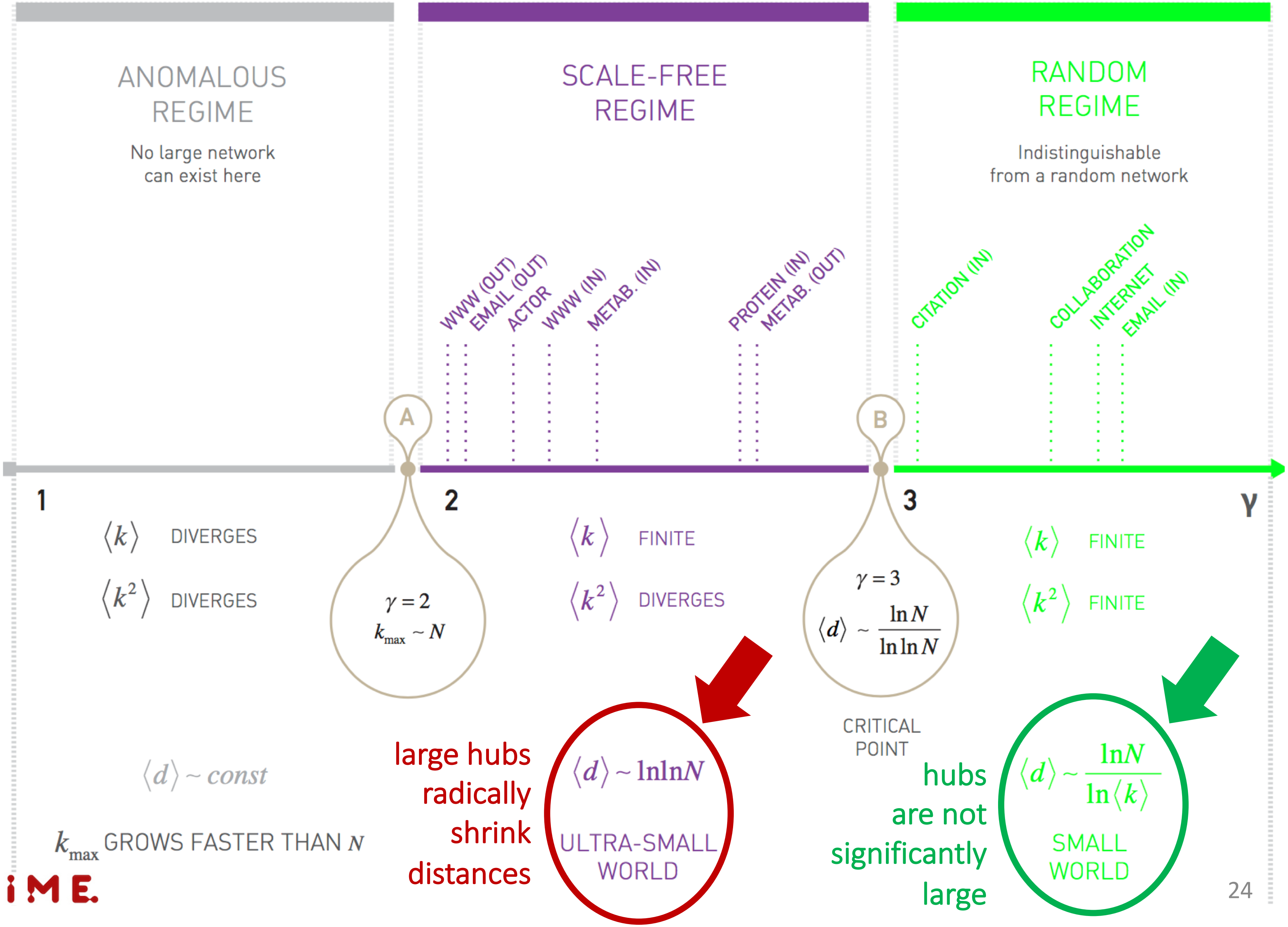
Moments of the power law $p_k = C \cdot k^{-\gamma}$

- $\langle k^n \rangle = \int_{k_{min}}^{k_{max}} k^n \, p_k \, dk$
  $= C \cdot (k_{max}^{n-\gamma+1} - k_{min}^{n-\gamma+1})/(n-\gamma+1)$
  $= C \, k_{min}^{n-\gamma+1} \cdot (N^{n/(\gamma-1)-1} - 1) / (n-\gamma+1)$

- They diverge with N if $\gamma < n+1$
  e.g. variance ($n=2$) diverges for $\gamma \in [2,3]$
  and the network does not have a scale



The scale-free regime

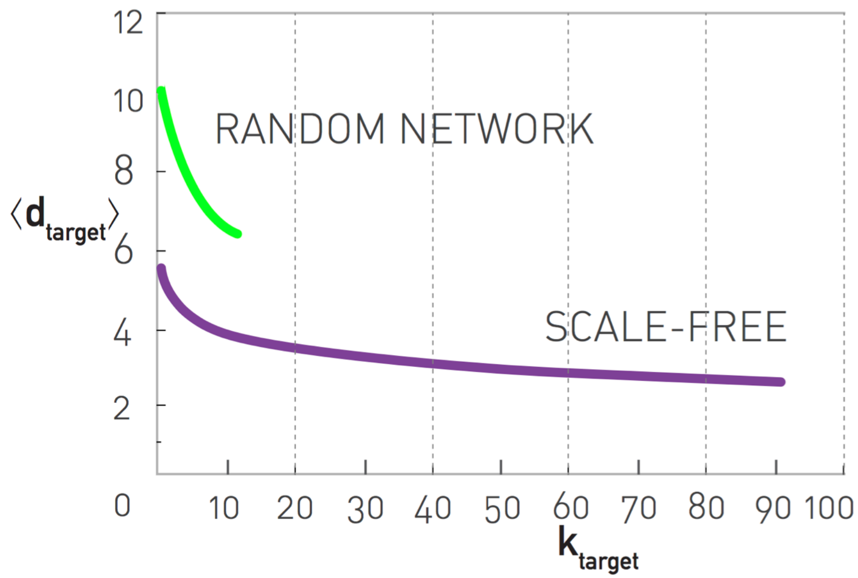


Ultra-small-world, 2<ɣ<3

- The average distance increases as ln(ln(*N*)), much slower than *N* or ln(*N*)

e.g. in www $N=7\cdot10^9$, ln(*N*)=*22.7*, ln(ln(*N*))=*3.12* (very small)

distance of a node of degree ⟨*k*⟩ from a node of degree k_{target}



- The large hubs radically shrink the distance between nodes → ultra small world

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Curiosity

In many social experiments people avoided hubs for entirely perceptual reasons (e.g., they assumed they are busy, better use them only if really needed)

We live in a ultra-small-world, but we perceive that we are more distant from others than we really are!

Friendship paradox

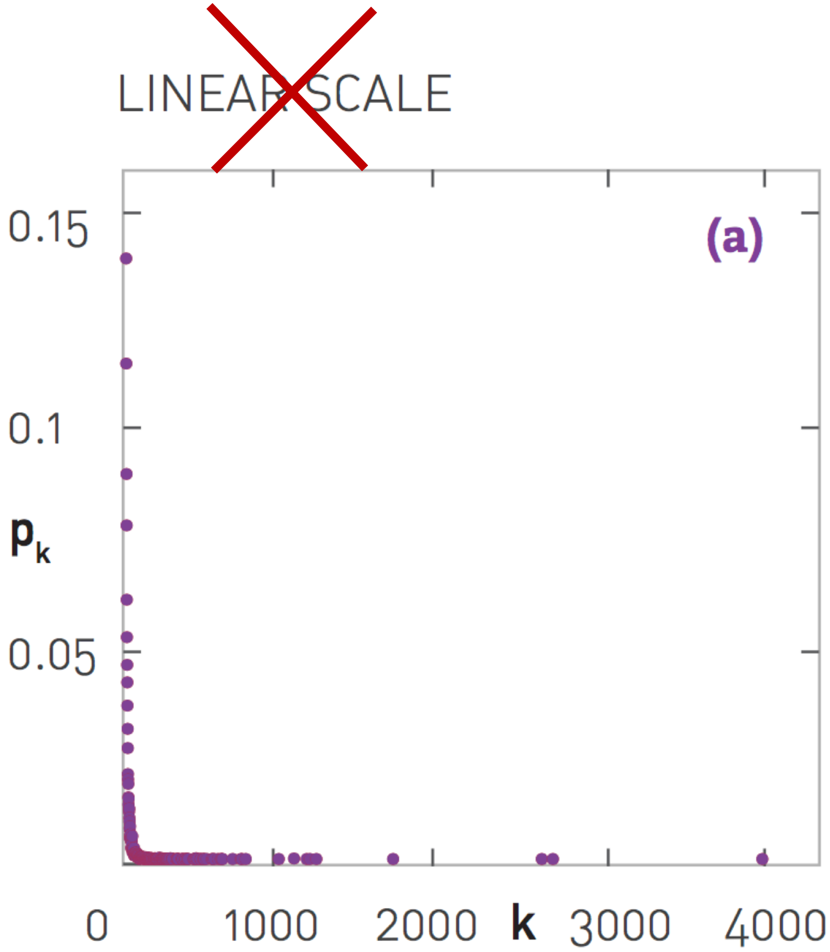
My friends are more popular than me! ☹

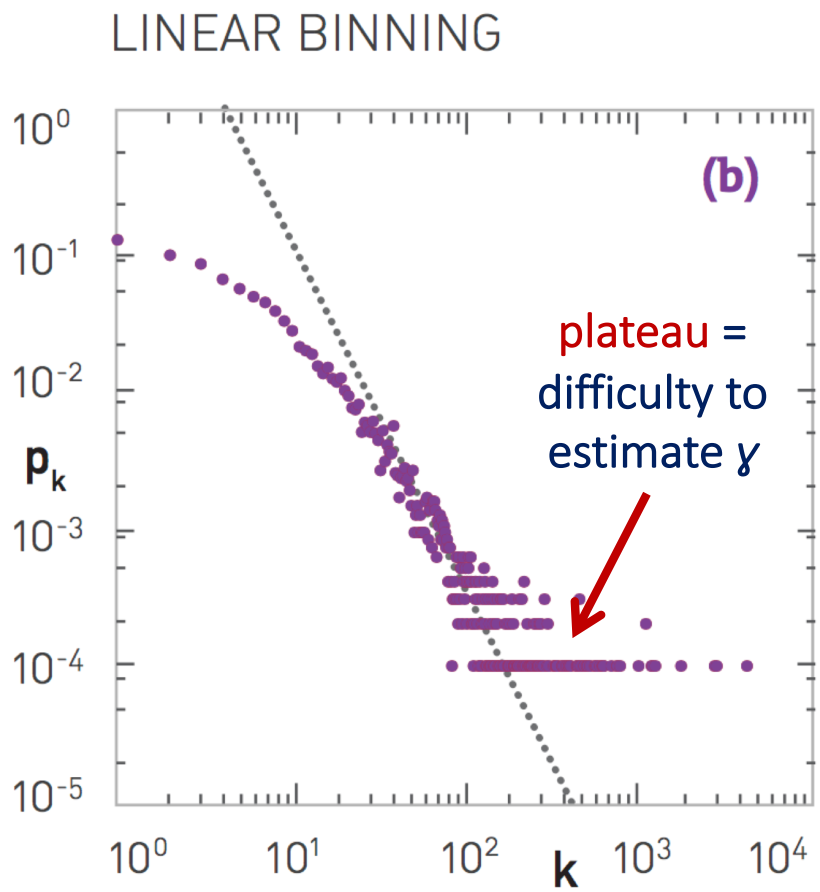
(Feld, 1991)

- Can be observed in the ultra-small-world under the presence of big hubs
- Rationale: a node is very likely to be connected to a big hub, having a very large number of connections
- # of friends (in the average) = $\langle k \rangle$
- # of friends of friends $\simeq N$

Estimating the exponent

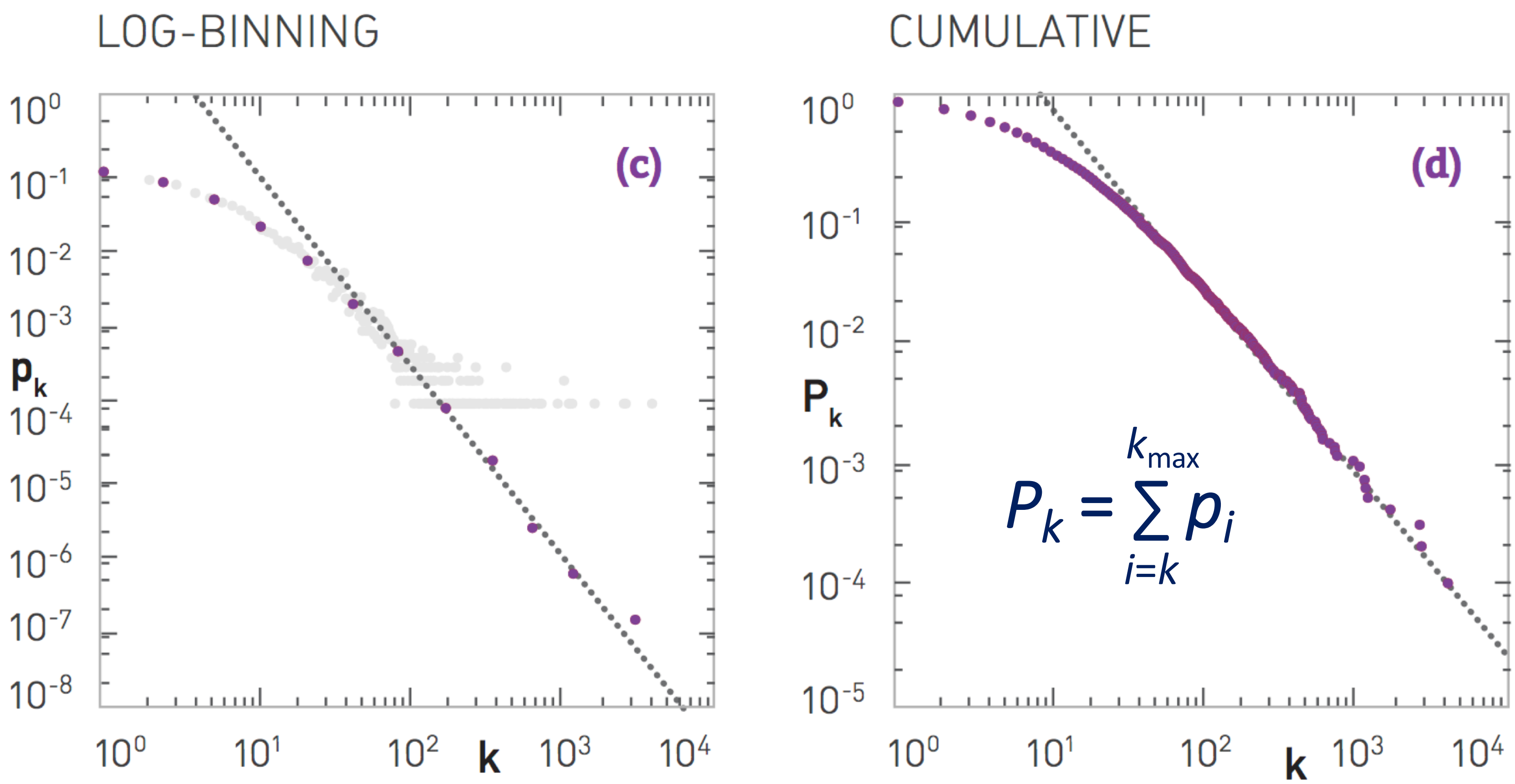
Plotting power laws





Better use a log-log scale

Plotting power laws (cont'd)



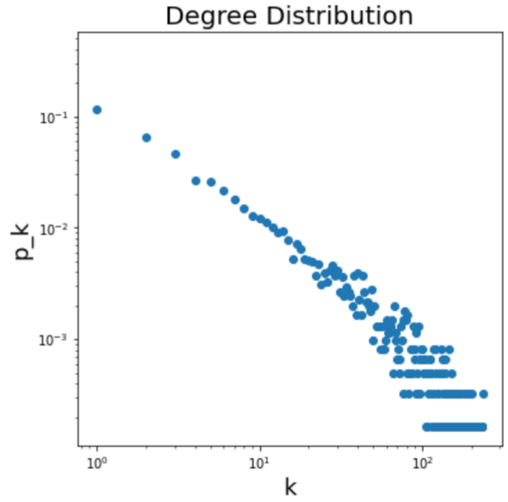
Even better: log-binning or complementary cumulative distr. function (CCDF) $P_k \sim k^{-(\gamma-1)}$

Pseudocode example

```
G = np.loadtxt('Wiki-Vote.txt').astype(int)

# adjacency matrix
N = np.max(G)
A = csr_matrix((np.ones(len(G)), (G[:, 1], G[:, 0])))


#distribution
which_deg = 0 # 0=out degree, 1=in degree
d = np.sum(A, which_deg) # out degree for each node
d = np.squeeze(np.asarray(d)) # from matrix to array
d = d[d>0] # avoid zero degree
k = np.unique(d) # degree samples
pk = np.histogram(d, k)[0] # occurrence of each degree
pk = pk/np.sum(pk) # normalize to 1
Pk = 1 - np.cumsum(pk) # complementary cumulative
```




```
fig = plt.figure()
plt.loglog(pk, 'o')
plt.title("Degree Distribution", size = 20)
plt.xlabel("k", size = 18)
plt.ylabel("p_k", size = 18)
plt.show()
```

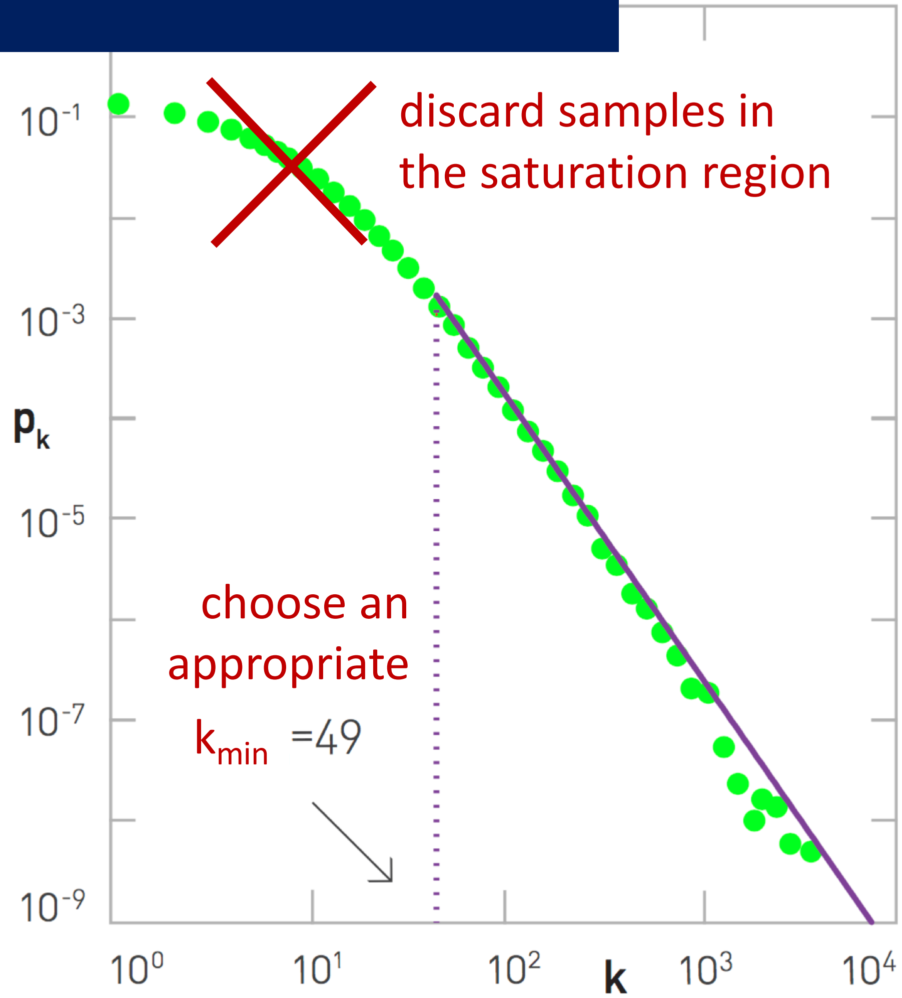
Estimating γ

Fit γ only in a wisely chosen interval

- $p_k = C\, k^{-\gamma}$
- $C = (\gamma-1)\, k_{min}^{\gamma-1}$

is determined by the normalization condition

$$\int_{k_{min}}^{\infty} p_k\, dk = C \cdot k_{min}^{-(\gamma-1)} / (\gamma-1) = 1$$



ML estimate for γ

- Target PDF $p(k|\gamma) = (\gamma\text{-}1)/k_{min} \cdot (k/k_{min})^{-\gamma}$
- ML criterion (k_i is the measured degree of node i)

$$\max_\gamma f(\gamma) = \sum_i \ln p(k_i|\gamma)$$

- $f(\gamma) = \sum_i \ln((\gamma\text{-}1)/k_{min}) - \gamma \ln(k_i/k_{min})$
- Solve $f'(\gamma) = \sum_i 1/(\gamma\text{-}1) - \ln(k_i/k_{min}) = 0$
- The result is

$$\gamma = 1 + \sum_i 1 / \sum_i \ln(k_i/k_{min})$$

Pseudocode example

```
which_deg = 1; % 1 = out degree, 2 = in degree
d = full(sum(A,which_deg));
d2 = d(d>=kmin); % restrict range
ga = 1+1/mean(log(d2/kmin)); % estimate the exponent
```

Other network models

Watts-Strogatz model [1998]

- It is the small-world model
  - ✓ Generalises the random network
  - ✓ It stresses the small-world property ☺
  - ✓ But predicts a Poisson like degree distribution ☹

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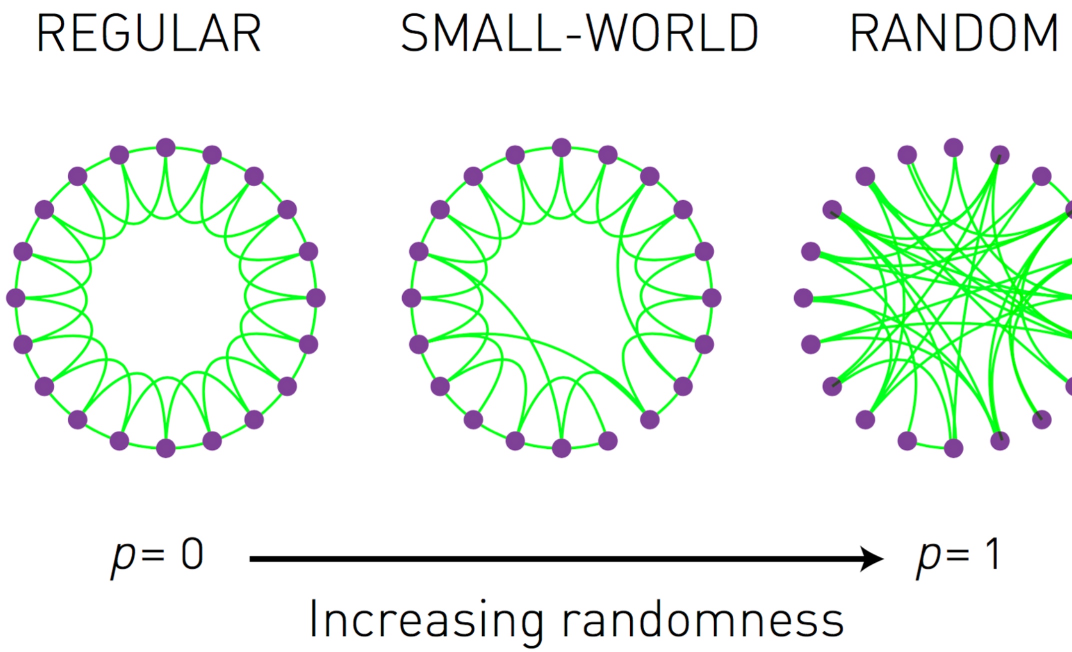
Watts-Strogatz model (cont'd)



- Start from a regular lattice
- Each node in the circle is connected to 2 neighbours on the right, and 2 on the left
- Rewire at random with probability *p*

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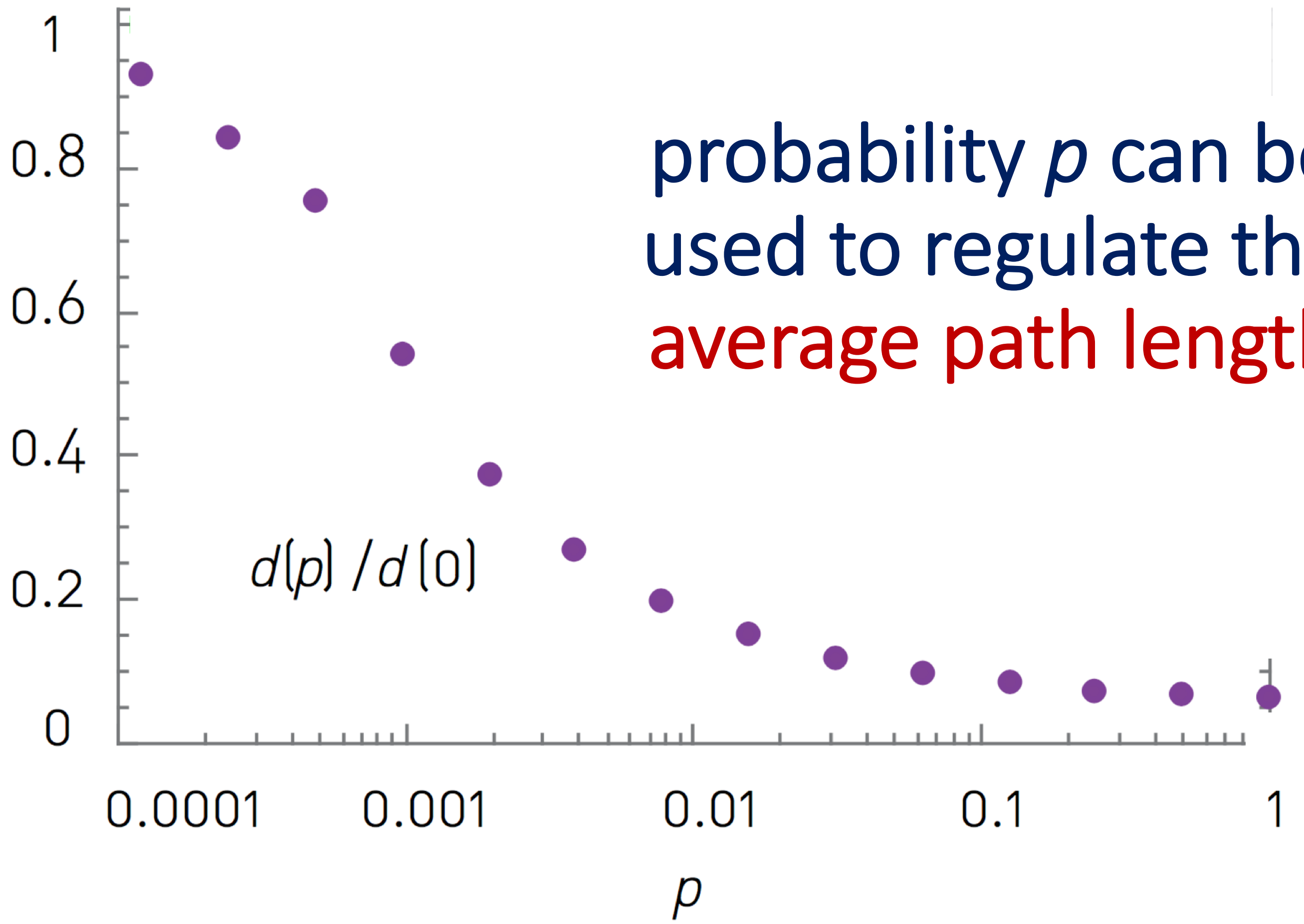
Watts-Strogatz model (cont'd)

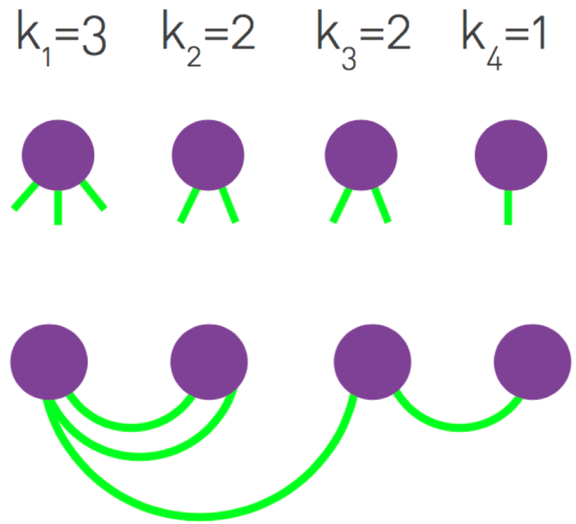


probability *p* can be used to regulate the **average path length**

Molloy-Reed model [1995]

1. assign a degree to each node
   make sure that the # of stubs is even, *2L*

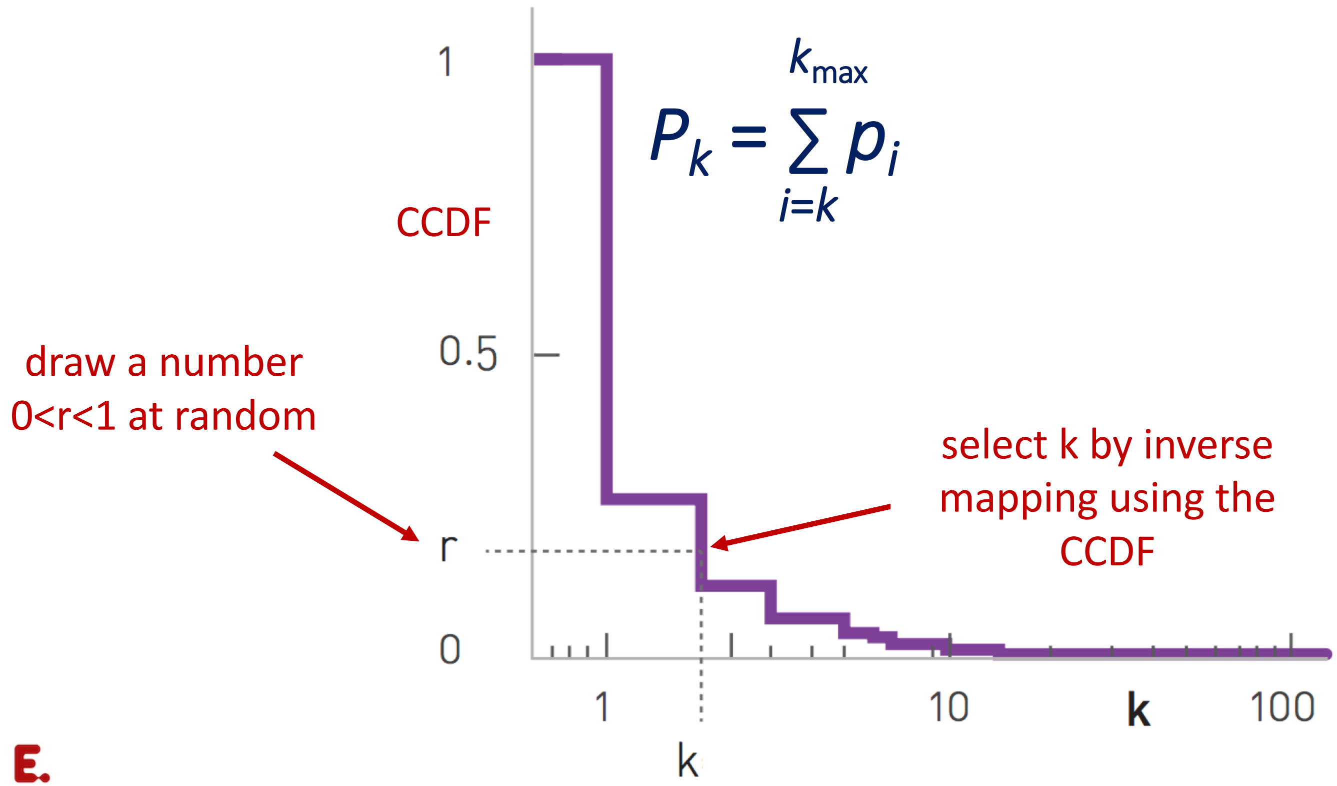
2. rewire stubs at random



- The resulting graph may contain cycles and multiple links (but are a few)
- ☹ cannot control features other than p_k

Molloy-Reed model [1995]

- How to generate degrees with given distribution p_k?
- Use the inverse CCDF method



Generalizations

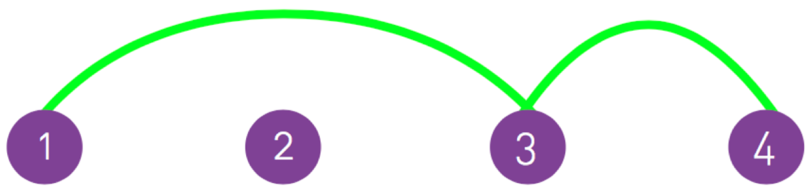
- Degree-preserving rewiring can help in reducing self/multiple-loops:
  1. select two links (T_1,S_1) and (T_2,S_2)
  2. rewire them as (T_1,S_2) and (T_2,S_1) = flip

- Can also be applied to directed networks by separately generating k_{in} and k_{out} stubs

Chung-Lu model

The hidden parameter model:

- activates a link with probability $p_{ij} = k_i k_j / \langle k \rangle N$



- prevents from having multi/self-loops
- ☹ has very high complexity $\propto N^2$
- ☹ cannot control features other than p_k
- can be extended to directed networks by considering $p_{i \to j} = k_i^{out} k_j^{in} / L$

Readings

- A.L. Barabási, Network science

  http://barabasi.com/networksciencebook

  Ch.3 “Random networks”

  Ch.4 “The scale-free property”